ASTR 425/525
Homework 1 solutions

Fall 2025

Due September 15th

Contents

Problem 1

a) If the typical total mass of a galaxy is about 10^{12} $M_\odot$, what is the average matter density of the Universe today, in $M_\odot/\text{Mpc}^3$, assuming that all matter in the Universe resides within galaxies.

b) How does that compare with the current density of the Universe, $\rho_c \simeq 1.4 \times 10^{11}$ $M_\odot/\text{Mpc}^3$?

a) We need to compute something of the form

$$\text{(Average) matter density of the universe} = \text{Typial mass} \cdot \text{Galaxy number density} \tag{1.1}$$

We are given that the typical (total) mass of a galaxy is about 10^{12} $M_\odot$. We still need to compute a galaxy number density[1] first.

Recall that (see the `Distances and the metric` lecture notes) the typical distance between large galaxies in our Universe is ~ 1 Mpc. It is then plausible to think that in a sphere with radius of half this distance contains, roughly, one galaxy. This gives a number density of

$$\begin{aligned} n_{\text{galaxy}} &= \frac{1}{V} \\ &= \frac{1}{\frac{4}{3}\pi r^3} \\ &= \frac{3}{4\pi(0.5)^3} \text{ Mpc}^{-3} \\ &= 1.90986 \text{ Mpc}^{-3} \end{aligned} \tag{1.2}$$

We can now compute that the average matter density is

$$\begin{aligned} \rho &= (\text{typical mass}) \cdot n_{\text{galaxy}} \\ &= 1.91 \times 10^{12} \, \frac{M_\odot}{\text{Mpc}^3} \end{aligned}$$

b) The computed value is 13.64 times larger than the current density of the universe. This difference can be attributed to the assumptions that we made. Furthermore, one can consider computing n_{galaxy} in slightly different ways.

■

[1] If we have a given volume in space, how many galaxies can we expect to find inside?

Problem 2

Our current universe appears to be dominated by a cosmological constant. Compute the age of our universe assuming that today (when the Hubble expansion rate is $H_0 = 70$km/s/Mpc) 70% of the energy is in the form of the cosmological constant and 30% is in the form of cold matter.

We are given the fractional densities of the species today, that is:

$$\Omega_\Lambda = 0.7 \tag{2.3}$$
$$\Omega_{\rm cdm} = 0.3 \tag{2.4}$$

both of which are related to the Hubble rate through the Friedmann equation (see Eqs. 2.141, 2.144 in Baumann)

$$H(a) = H_0\sqrt{\Omega_{\rm cdm}a^{-3} + \Omega_\Lambda} \tag{2.5}$$

By recalling the definition of the Hubble rate, $H \equiv \dot{a}/a = \frac{1}{a}\frac{da}{dt}$, we can rewrite Eq. 2.5 as an explicit differential equation for $a(t)$:

$$H = \frac{1}{a}\frac{da}{dt} = H_0\sqrt{\Omega_{\rm cdm}a^{-3} + \Omega_\Lambda} \tag{2.6}$$

And we can rearrange to isolate the variables a, t on their respective sides of the equality:

$$dt = \frac{1}{aH_0\sqrt{\Omega_{\rm cdm}a^{-3} + \Omega_\Lambda}}da \tag{2.7}$$

The age of the universe can then be computed by integrating Eq. 2.7:

$$t = \int_0^1 \frac{1}{aH_0\sqrt{\Omega_{\rm cdm}a^{-3} + \Omega_\Lambda}}da \tag{2.8}$$

Where the t integral goes from 0 to t. This integral can be solved analytically (see the hint on worksheet # 7). Still, it is good to know how to evaluate integrals numerically, and this is a good simple example to learn from. Find below python code that evaluates and prints the resulting value of this integral.

```python
import numpy as np
import scipy.integrate as integrate

# Set universe parameters
Omega_Lambda = 0.7
Omega_cdm = 0.3
H0 = 70 # km/sec/Mpc

# Define the function we want to integrate
def integrad(a):
    return 1/(a*H0*np.sqrt(Omega_Lambda+Omega_cdm*a**(-3)))

# Integrate
# Observe that the integrate.quad() function has 3 parameters here:
# The integrand and the lower/upper limits of integration.
age_of_the_universe, error = integrate.quad(integrad, 0, 1)
print(age_of_the_universe, "(km/s/Mpc)^-1")
```

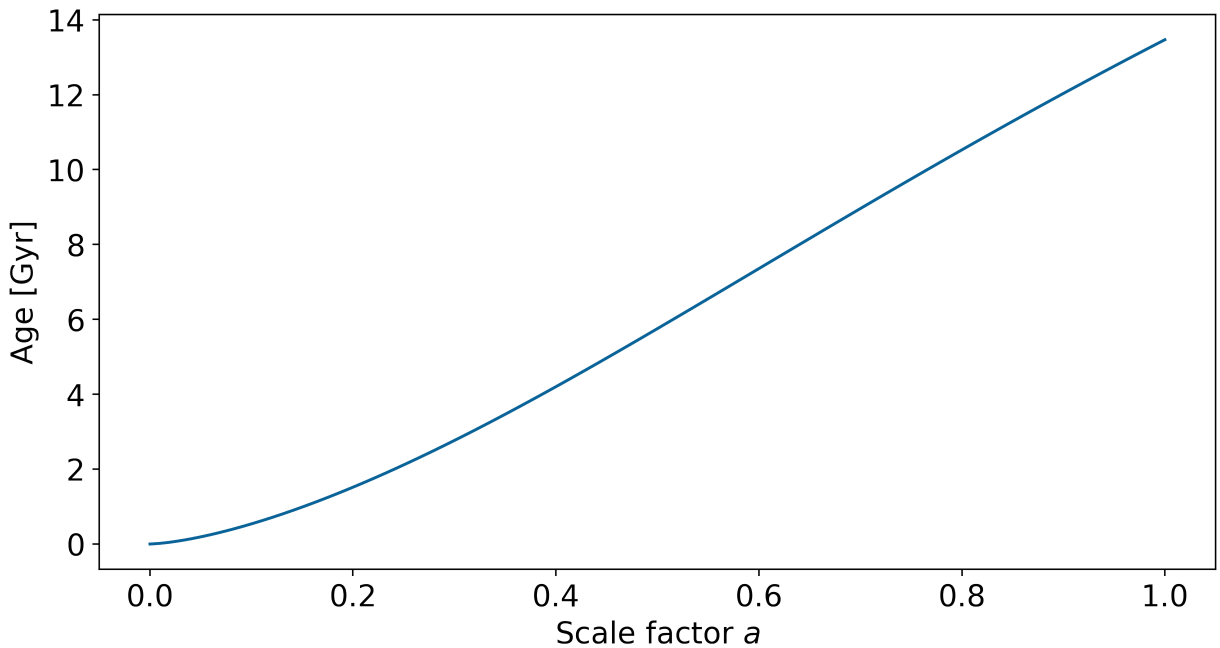
Returning

$$\text{age of the universe} = 1.37728 \times 10^{-2}\ \text{s}\frac{\text{Mpc}}{\text{km}} \tag{2.9}$$

Observing that there are 3.086×10^{19} km in a Mpc, and that there are 31556952 seconds in a year, we see that

$$\begin{aligned} \text{age of the universe} &= 1.37728 \times 10^{-2}\ \text{s}\frac{\text{Mpc}}{\text{km}} \cdot \frac{3.086 \times 10^{19}\ \text{km}}{1\ \text{Mpc}} \cdot \frac{1\ \text{yr}}{31556952\ \text{s}} \\ &= 1.34686 \times 10^{10}\ \text{yr} \\ &= \boxed{13.468\ \text{Gyr (billions of)}} \end{aligned} \tag{2.10}$$

■



(Plot not required for submission)
Cosmic age as a function of scale factor.

The analytic solution to the integral is:

$$\int_0^1 \frac{da}{aH\sqrt{\Omega_{\text{cdm}}a^{-3} + \Omega_\Lambda}} = \frac{2}{3H\sqrt{\Omega_\Lambda}} \arcsin\left(\sqrt{\frac{\Omega_\Lambda}{\Omega_{\text{cdm}}}}\right) \tag{2.11}$$

Problem 3

a) As you saw in the previous problem, the age of the Universe is tightly related to the Hubble constant H_0. However, in the far future, once the cosmological constant *completely* dominates the energy density of our universe, this relationship will break down. Show that a far-future civilization will determine the age of the Universe to be infinite, independent of H_0.

b) What does this tell you about the symmetry structure of this Universe? Does it have more symmetry as compared to a standard FLRW universe dominated by nonrelativistic matter?

a) Consider the far future when the cosmological constant energy density completely dominates. By neglecting any other component ($\Omega_{\mathrm{cdm}}, \Omega_{\mathrm{b}}, \Omega_{\mathrm{r}}, \Omega_k \to 0$, making $\Omega_\Lambda = 1$ by definition), we observe that the Hubble rate becomes constant:

$$\begin{aligned} H(a) &= H_0\sqrt{\Omega_\Lambda + \text{negligible components}} \\ &= H_0\cancel{\sqrt{1}} \\ &= H_0 \end{aligned} \tag{3.12}$$

Following the same process as in problem 2, we can conclude that the age of the universe will be given by:

$$\begin{aligned} t &= \frac{1}{H_0}\int_0^1 \frac{da}{a} \\ &= \frac{1}{H_0}\Big(\log(1) - \log(0)\Big) \\ &= \frac{1}{H_0}(0 - -\infty) \\ &= +\infty \end{aligned} \tag{3.13}$$

b) The universe acquires a temporal symmetry, making it (in the limit of a far future, i.e., asymptotically) time-*translation* invariant (analogous to isotropy/homogeneity for spatial symmetries).

We won't be able to *distinguish* different times, because the universe will look the same whether we look at it now or later in time (the energy composition remains fixed and the Hubble rate becomes constant).

This kind of scenario is interesting for many reasons, such as (but not limited to):

- The expansion of the universe ($a(t)$) is a constant-rate exponential that won't slow down.
- The horizon size is constant (here we mean the physical/proper distance) at all times.
- Observers born into this universe won't be able to tell much about their cosmological past. Any kind of useful information red-shifted/moved outside their horizon.
- These kinds of universes are examples of *de Sitter* spaces. There are some interesting properties about these spaces, such as a non-zero vacuum temperature.

Problem 4

Let's consider the flat FLRW metric:

$$ds^2 = -dt^2 + a^2(t)\left[dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\phi^2)\right]$$

where $a(t)$ is the scale factor. For this problem, assume a realistic universe filled with matter, radiation, and a cosmological constant, and use the following values of the cosmological parameters:

$$\begin{aligned} H_0 &= 70.4 \text{ km/s/Mpc} \\ \Omega_r &= 0.000084 \\ \Omega_m &= 0.272 \\ \Omega_\Lambda &= 0.728\ . \end{aligned}$$

Here, we shall refer to epochs in the evolution of the Universe in terms of their *redshift* z, which is related to the scale factor by

$$a(t) = \frac{1}{1+z}$$

a) Using the fact that photons always travel on null trajectories ($ds^2 = 0$), compute the total comoving distance that a photon will travel from the Big Bang at $t = 0$ to the epoch of recombination at redshift $z = 1090$.

b) Now compute the total comoving distance that a photon will travel from the epoch of recombination to the present time ($z = 0$).

c) Divide your answer from part (a) by that from part (b). Using a diagram, show that this ratio is the maximum angle separating photons that were in causal contact in the distant past, according to the metric above. Express this angle in degrees. Now, we observe that cosmic microwave background (CMB) photons (which were emitted at the epoch of recombination at $z = 1090$) from opposite points in the sky (i.e., points separated by 180 degrees) to have the same temperature to a high accuracy. Do you see a problem here? Explain.

a) We want to find the comoving distance (i.e. distance on the expanding-grid of the universe) for a photon traveling ever since the big bang up until redshift $z = 1090$ (corresponding to the epoch of recombination). Because of the null trajectory ($ds^2 = 0$), we see that

$$dt^2 = a^2(t)dr^2 \tag{4.14}$$

So the comoving distance is:

$$r_{\text{com}} = \int dr = \int_{t=0}^{t_{\text{rec}}} \frac{1}{a(t)} dt \tag{4.15}$$

Observe that we not only don't know (yet) what the integrand is (how does a depend on t), but we also don't know the equivalent cosmic time t corresponding to $z_{\text{rec}} = 1090$. The easiest way to approach this is to rewrite the t integral as an integral in terms of a (or equivalently, z). Recall that we can rewrite the Friedmann equation as (see Eq. 2.144 in Baumann)

$$H(a) = H_0\sqrt{\Omega_{\text{r}} a^{-4} + \Omega_{\text{m}} a^{-3} + \Omega_{\text{k}} a^{-2} + \Omega_\Lambda} \tag{4.16}$$

Since $H = \dot{a}/a = \frac{da}{dt}\frac{1}{a}$, it follows that:

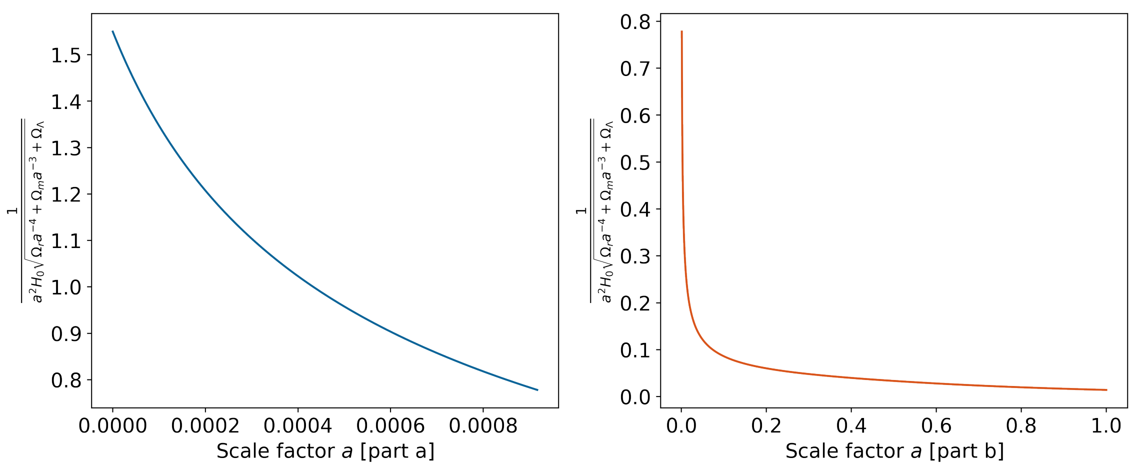
$$dt = \frac{da}{Ha} \tag{4.17}$$

So Eq. 4.15 can be written as

$$\begin{aligned} r_{\text{com}} &= \int_0^{t_{\text{rec}}} \frac{dt}{a} \\ &= \int_0^{a_{\text{rec}}} \frac{da}{a^2 H} \\ \text{(setting } \Omega_{\text{k}} = 0) \qquad &= \int_0^{a_{\text{rec}}} \frac{da}{a^2 H_0 \sqrt{\Omega_{\text{r}} a^{-4} + \Omega_{\text{m}} a^{-3} + \Omega_\Lambda}} \end{aligned} \tag{4.18}$$

This integral can't be solved analytically, so we must resort to numerical methods. Because of its relatively short integration interval, this integral is not inherently numerically-unstable. Nonetheless, it is good practice to compute these integrals over redshift z rather than scale factor a. Not only will the integrand be simpler, but it will make the integral numerically-tractable for longer integration intervals such as the one in part (b).

Why changing to z is important in most cases.



One should generally plot integrands like the one on the right using a logarithmically spaced horizontal axis. Here, a linear axis was used deliberately to illustrate potential numerical instabilities. Observe that, over the integration domain for part (b), the integrand varies rapidly at small a, to then become relatively stable at larger a. Most conventional numerical integration codes/routines may struggle to find a reliable value if such the integrand is not handled carefully and without human-set precision parameters.

Recalling that

$$a = \frac{1}{1+z} \tag{4.19}$$

We observe that

$$da = -\frac{dz}{(1+z)^2} = -a^2 dz \tag{4.20}$$

And the integral in Eq. 4.18 becomes

$$r_{\text{com}} = \int_{z_{\text{rec}}}^{\infty} \frac{dz}{H_0\sqrt{\Omega_{\text{r}}(1+z)^4 + \Omega_{\text{m}}(1+z)^3 + \Omega_\Lambda}} \tag{4.21}$$

Where the negative sign was *absorbed* while flipping the limits of integration. Find code below:

```
import numpy as np
import scipy.integrate as integrate

# Set universe parameters
H0 = 70.4 # km/s/Mpc
Omega_r = 0.000084
Omega_m = 0.272
Omega_L = 0.728

# Define the function we want to integrate
def integrad(z):
    return 1/(H0*np.sqrt(
        + Omega_r*(1+z)**4
        + Omega_m*(1+z)**3
        + Omega_L)
            )

z_rec = 1090
z_big_bang = np.inf

c = 2.997e8 # m/s. To conver from (km/s/Mpc)^-1 to Mpc
# Integrate
r_com, error = integrate.quad(integrad, z_rec, z_big_bang)
print(f"{r_com:.4e} (km/s/Mpc)^-1")
print(f"{r_com*c/1000:.4e} Mpc") # The factor of 1/1000 converts km to m
```

From which we find that

$$r_{\text{com}} = 2.8459 \times 10^2 \text{ Mpc} \tag{4.22}$$

b) We can recycle the equation/code from part (a). We just need to modify the integration limits. The initial time is $z_{\text{rec}} = 1090$ and the final time is $z_{\text{today}} = 0$ (minding the order in which we place these limits of integration)

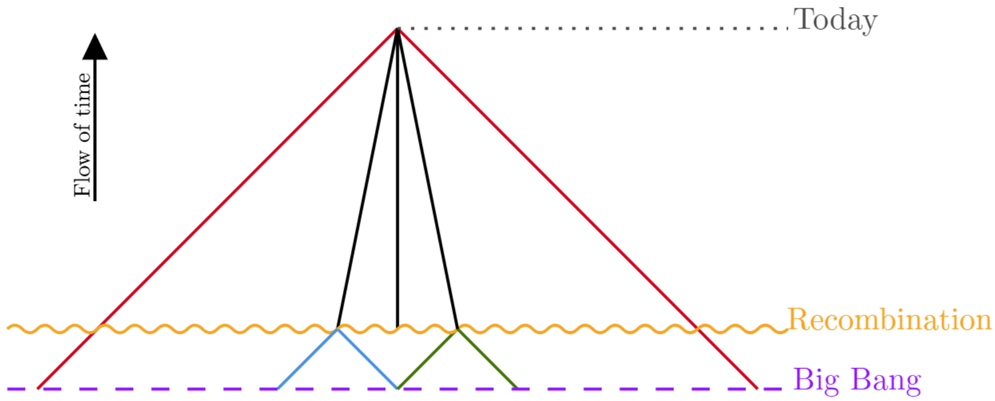
```
r_com, error = integrate.quad(integrad, 0, z_rec)
print(f"{r_com:.4e} (km/s/Mpc)^-1")
print(f"{r_com*c/1000:.4e} Mpc") # The factor of 1/1000 converts km to m
```

We find

$$\begin{aligned} r_{\text{com}} &= 1.4072 \times 10^4 \text{ Mpc} \\ &= 14.072 \text{ Gpc} \end{aligned} \tag{4.23}$$

Fun fact: If we print the variable `error` returned by `integrate.quad()`, we can observe that the *integration error* computed using the integrand expressed in scale factor is roughly 2.55 times larger than the error computed using the redshift version of the integrand.

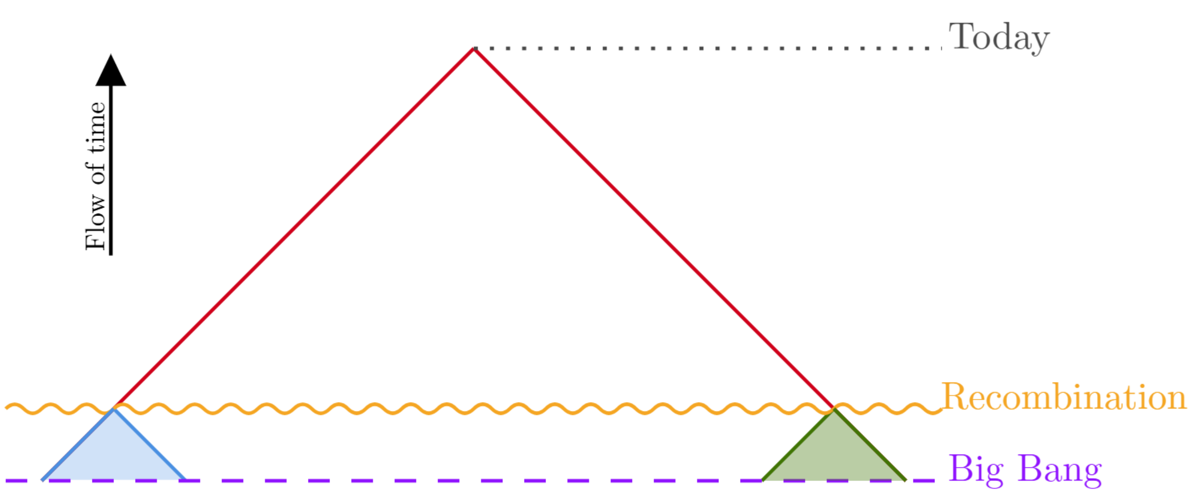
c) What we realize here is that the comoving distance a photon travels between the big bang and the time of recombination is much (very much) smaller than the comoving distance that a photon travels between the time of recombination and today.



Our past light cone (in red) sets the boundaries of what we can see. This light cone is quite large, but notice that can observe light whose past light cones never intersect, hence they never have causal contact. There is a maximum angle (spanning the black triangle) at which we can observe photons from the CMB whose light cones intersect (here in blue and green). This angle comes out to be

$$\begin{aligned} \theta &\approx \frac{284.59}{14072} \text{ Radians} \\ &= 2.02238 \times 10^{-2} \text{ Radians} \\ &= 1.158^\circ \end{aligned} \tag{4.24}$$

If we are observing that opposite points in the sky have the same temperature, then there is something interesting going on. Why do they have the same temperature if these points were never in causal contact? This is known as the *horizon problem*.



The problem is not that we can see light coming from the boundary of our past light cone, but rather that their own light cones don't intersect even though our observations seem to suggest they do. ∎

Problem 5

Imagine a spatially-flat homogeneous and isotropic expanding universe filled *exclusively* with a fluid with an equation of state $w = 1$ such that $p = \rho$. Let's denote the present-day Hubble expansion rate as H_0 in this universe.

a) Using conservation of energy in an expanding universe, show that the energy density of this fluid scales with the scale factor of the Universe $a(t)$ as

$$\rho \propto a(t)^{-6}.$$

b) Compute the present age of this universe t_0, as a function of H_0. Is this universe older or younger than a spatially-flat universe entirely filled with non-relativistic matter ($w = 0$) with the same Hubble constant today?

c) Now imagine that after greatly improving their cosmological measurements, the inhabitants of this universe discover that their universe is in fact *not* spatially flat but has instead a small spatial curvature given by

$$\Omega_K = -0.01 \ .$$

Argue that this implies that their universe will eventually stop expanding and start to contract. Assuming the usual normalization of the scale factor such that $a(t_0) = 1$, compute the value of the scale factor at which this turn-around point occurs.

d) Compute the total age t_{BC} that this universe have at the time of its "big crunch" when the entire universe has collapsed back to a point (i.e., $a(t_{\mathrm{BC}}) = 0$).

a) Let's formally solve the continuity equation (Eq. 7, in the `Continuity equation` lecture notes):

$$\begin{aligned}
\frac{d\rho}{dt} + 3\frac{da}{dt}\frac{\rho + p}{a} &= 0 \\
\frac{d\rho}{da}\frac{\cancel{da}}{\cancel{dt}} + 3\frac{\cancel{da}}{\cancel{dt}}\frac{2\rho}{a} &= 0 \\
\frac{d\rho}{\rho} &= -6\frac{da}{a} \\
\int \frac{d\rho}{\rho} &= \int -6\frac{da}{a} \\
\ln\rho &= -6\ln a + C \\
\ln\rho &= \ln a^{-6} + C \\
\rho &\propto a^{-6}
\end{aligned} \tag{5.25}$$

Where the proportionality is taken to drop the constant term. Observe that this proportionality becomes an equality by introducing an *initial condition*, namely:

$$\rho(a) = \rho_0 a^{-6} \tag{5.26}$$

b) Recall that

$$H \equiv \frac{\dot{a}}{a} = \frac{da}{dt}\frac{1}{a} \tag{5.27}$$

So

$$dt = \frac{da}{aH} \tag{5.28}$$

Where, in this universe filled exclusively with a $w = 1$ fluid,

$$\begin{aligned} H(a) &= H_0\sqrt{\Omega_{\text{fld}} a^{-6}} \\ &= H_0 a^{-3} \end{aligned} \tag{5.29}$$

The age of this universe follows as

$$\begin{aligned} t_{\text{age}} &= \int_0^t dt \\ &= \int_{a=0}^1 \frac{da}{aH_0 a^{-3}} \\ &= \frac{1}{H_0}\int_0^1 a^2 da \\ &= \frac{1}{H_0}\cdot\frac{1}{3} \\ &= 1/3\ H_0^{-1} \end{aligned} \tag{5.30}$$

A universe filled with non-relativistic matter is twice as old:

$$\begin{aligned} t_{\text{age}} &= \int_0^t dt \\ &= \int_{a=0}^1 \frac{da}{aH_0 a^{-3(1+w)/2}} \\ \text{setting } w = 0... \qquad &= \frac{1}{H_0}\int_0^1 a^{1/2} da \\ &= 2/3\ H_0^{-1} \end{aligned} \tag{5.31}$$

c) If $\Omega_K = -0.01$, then we must have

$$\Omega_{\text{fld}} = 1.01 \tag{5.32}$$

From the Friedmann equation we have

$$\frac{da}{dt} = H_0 a\sqrt{\Omega_{\text{fld}} a^{-6} + \Omega_K a^{-2}} \tag{5.33}$$

The "turn-around" happens when $da/dt = 0$. This condition is equivalent to

$$\begin{aligned} \Omega_K + \Omega_{\text{fld}} a^{-4} &= 0 \\ \frac{1}{a^{-4}} &= \frac{\Omega_{\text{fld}}}{-\Omega_K} \\ a^4 &= \frac{1.01}{0.01} \\ a &= 3.17015 \end{aligned} \tag{5.34}$$

So when the universe expands to a size 3.17 times today's size (with normalization $a_{\text{today}} = 1$), the rate of change of the scale factor drops to 0. This is the turning point.

d) The time to reach the turning point is:

$$\begin{aligned} t_{\max} &= \frac{1}{H_0} \int_{a=0}^{a_{\max}} \frac{da}{a\sqrt{\Omega_{\text{fld}} a^{-6} + \Omega_K a^{-2}}} \\ &= 18.9564 \; H_0^{-1} \end{aligned} \tag{5.35}$$

The *lifespan* of the universe (the time it takes for it to grow from $a = 0$ to its largest expansion $a_{\max}$ and back) is then given by twice the time it takes to reach the maximum:

$$t_{\text{BC}} = 37.9127 \; H_0^{-1} \tag{5.36}$$

■