ASTR 425/525
Homework 2 solutions

Fall 2025

Due September 29th

Contents

Problem 1 (2 points)

Show that for close-by objects at a distance d away from us ($H_0 d \ll 1$), the redshift z is approximately given by

$$z \simeq H_0 d \tag{1}$$

Does it matter whether d is a comoving, angular diameter, or luminosity distance?

Consider a Taylor expansion of $a(t)$ around today's time t_0:

$$a(t) \simeq a(t_0) + \dot{a}(t_0)(t - t_0) + \mathcal{O}\left[(t - t_0)^2\right] \tag{2}$$

Where we can ignore terms of order 2 and higher. Now divide both sides by $1 = a(t_0)$:

$$\begin{aligned} a(t) &= 1 + \frac{\dot{a}(t_0)}{a(t_0)}(t - t_0) \\ &= 1 + H_0(t - t_0) \end{aligned} \tag{3}$$

Then by definition of z:

$$\begin{aligned} z &= \frac{1}{a} - 1 \\ &\simeq \frac{1}{1 + H_0(t - t_0)} - 1 \\ &\simeq 1 - H_0(t - t_0) - 1 \\ &= H_0(t_0 - t) \end{aligned} \tag{4}$$

Where we used the approximation $\frac{1}{1+x} \simeq 1 - x$ (for small x). In units of $c = 1$ ($t - t_0 = d$) or with the appropriate factors of c included, we can conclude that at small scales

$$z \simeq H_0 d \tag{5}$$

This argument is reflected directly if we try to compute the comoving distance (relevant factor of c added for reference)

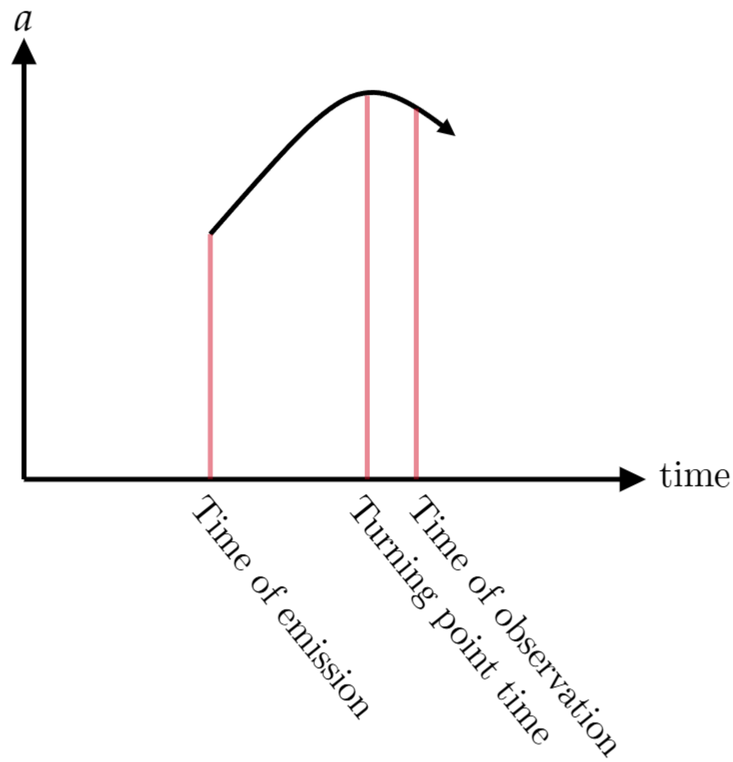
$$\begin{aligned} dr &= \frac{dt}{a(t)} \\ \chi &= c \int_{t_1}^{t_0} \frac{dt}{a(t)} \\ &\simeq c \int_{t_1}^{t_0} 1 - H_0(t - t_0)\, dt \\ \text{(Dropping 2nd order terms)} \quad &\simeq c(t_0 - t_1) \\ \text{(Using Eq. 4)} \quad &= c\frac{z}{H_0} \end{aligned} \tag{6}$$

This result fits the proposed relation between z and d (Eq. 1), where d is comoving distance here. At small scales, $S_k(\chi) = \chi$, and $a \simeq 1$, so d_A (the angular diameter distance) and d_L (the luminosity distance) are also well approximated by Eq. 1.

Problem 2 (1 points)

A galaxy emits light of a particular wavelength. As the light travels, the expansion of the Universe slows down and stops. Just after the Universe begins to recollapse, the light is received by an observer in another galaxy. Does the observer see the light redshifted or blueshifted?

The question of whether light is observed to be red or blueshifted is not about the current expansion rate but rather the current state compared to the time of emission.



Light is redshifted if a_{today} is larger than a_{emission}. Light is blueshifted if a_{today} is smaller than a_{emission}.

Since we are observing the galaxy light a bit after the turning point of the universe, it is safe to assume that $a_{\text{today}} > a_{\text{emission}}$, so we observe **redshifted light**.

Problem 3 (4 points)

- In a flat spacetime, objects of a fixed physical size subtend smaller and smaller angels as they are further and further away; in an expanding universe this is not necessarily true. Consider the angular size $\theta(z)$ of an object of physical size L at redshift z. In a spatially-flat universe with $\Omega_\mathrm{m} = 0.3$ and $\Omega_\Lambda = 0.7$, at what redshift is $\theta(z)/L$ minimum?
- Assuming $H_0 = 70$ km/s/Mpc, what is the angular size of a galaxy of physical size 10 kpc at this redshift?
- What is the angular size of this galaxy at redshift $z = 10$?

- Recall that

$$d_A \equiv \frac{l}{\theta} \tag{7}$$

Where θ is the angular size, $l \equiv L$, and d_A is the angular diameter distance. The quantity of interest follows as $\theta(z)/L = 1/d_A$. The angular diameter distance in a spatially flat universe is related to the comoving distance χ by

$$d_A(a) = a\chi \tag{8}$$

And

$$\chi(z) = \int_0^z \frac{dz'}{H(z')} \,. \tag{9}$$

In a universe with only matter and dark energy,

$$H(z) = H_0\sqrt{\Omega_\mathrm{m}(z+1)^3 + \Omega_\Lambda} \tag{10}$$

So

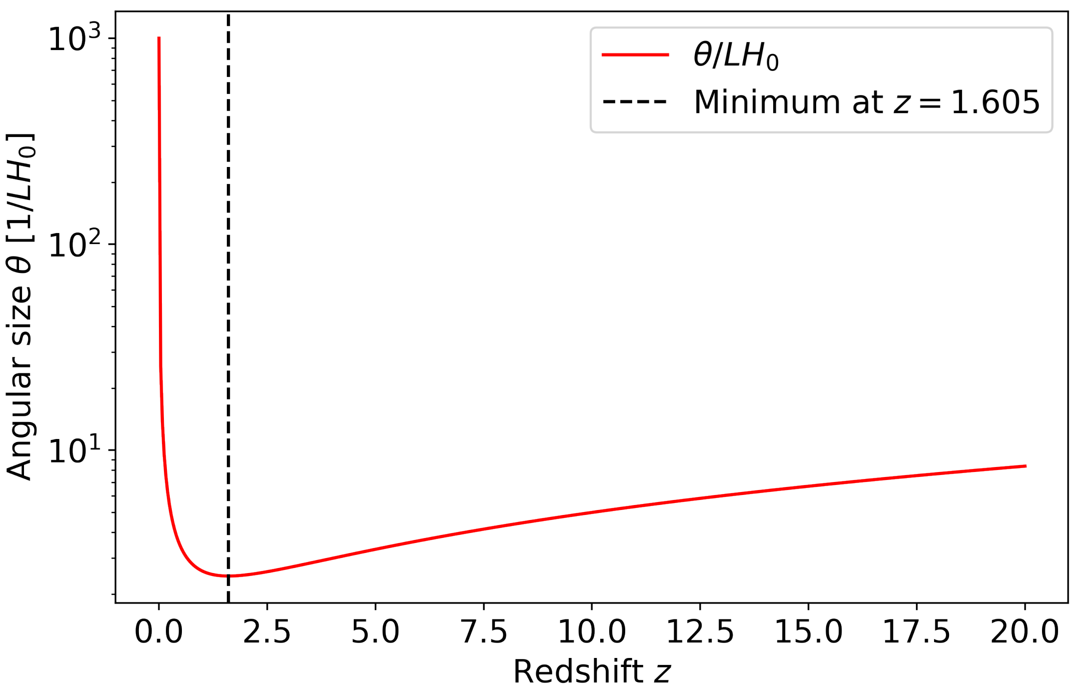
$$\begin{aligned} \theta(z)/L &= \frac{1}{a\chi(z)} \\ &= \frac{1+z}{\chi(z)} \\ &= \frac{1+z}{\int_0^z \frac{dz'}{H_0\sqrt{\Omega_\mathrm{m}(z'+1)^3+\Omega_\Lambda}}} \end{aligned} \tag{11}$$

We can use numerical methods to find the the minimum of this function. For this example, I use `minimize_scalar` (from `scipy.optimize`), but there are other libraries and approaches (e.g., one can consider finding $\theta'(z)$ and use a root-finding algorithm on this function instead.)

Find code and a plot on the next page.

```python
import matplotlib.pyplot as plt
import numpy as np
import scipy.integrate as integrate
from scipy.optimize import minimize_scalar

# Parameters
Omega_m = 0.3 # Matter
Omega_L = 0.7 # Dark Energy in the form of a cosmological constant.

def H(z):
    return np.sqrt(Omega_m*(1+z)**3 + Omega_L)

def integrand(z):
    return 1/H(z)

def chi(z):
    comoving_distance, integration_error = integrate.quad(integrand,0,z)
    return comoving_distance

def angular_size(z):
    return (1+z)/chi(z)

# Numerically find where the minimum occurs
min_z = minimize_scalar(angular_size,bounds=(0.001,5))
# Do note that the minimum is accessed by printing min_z.x, rather than min_z
print(f"The minimum angular size occurs at z = {min_z.x:.3f}")
```

```
The minimum angular size occurs at z = 1.605
```



Find code to reproduce this plot below.

```python
# Plot of the angular size
fig, ax = plt.subplots(figsize=(8, 5))
ax.tick_params(axis='both', labelsize=15)
Z = np.linspace(0.001,20,500)
plt.plot(Z,[angular_size(ZZ) for ZZ in Z],label=r"$\theta / LH_0 $",color="
    red")
plt.yscale("log")
plt.xlabel(r"Redshift $z$",fontsize=15)
plt.ylabel(r"Angular size $\theta$ [$1 /L H_0 $]",fontsize=15)
plt.axvline(x=min_z.x,color="black",label=rf"Minimum at $z=${min_z.x:.3f}",
    linestyle="--")
plt.legend(fontsize=15)
plt.savefig("problem_3_angular_size_as_a_function_of_z.png",dpi=400,
    transparent=True,bbox_inches='tight')
```

- Observing that

$$\theta(z) = \frac{L(1+z)}{\chi(z)} \tag{12}$$

We can compute χ using H_0 in km/s/Mpc as long as we handle the units accordingly. θ is a dimensionless number (radians), so all we have to take care of is having L in Mpc and adding a factor of c (speed of light) where appropriate to vanish km/s.

$$\begin{aligned}\theta &= (0.01\text{ Mpc})(1+z)\left(\chi(z)\text{ km/s/Mpc}^{-1}\right)^{-1}\left(\text{c}^{-1}\,\frac{\text{s}}{\text{m}}\right)\left(1000\,\frac{\text{m}}{\text{km}}\right) \\ &= 1.179\text{ arcseconds}\end{aligned} \tag{13}$$

With code:

```python
L =0.01 # Mpc
c = 3e8 # m/s
meters_over_kilometers = 1000
H0 = 70 # km/s/Mpc
redshift = min_z.x

radians = L*(1+redshift)*H0*(1/chi(redshift))*(1/c)*meters_over_kilometers
#                         ^
#                        This factor of H0 goes here since the chi function
#                        returns chi in units of [H0^-1]
degrees = radians*180/np.pi
arc_minutes = degrees * 60
arc_seconds = arc_minutes * 60
print(f"{arc_seconds:.3f} Arcseconds")
```

Note: Adding factors of c is not artificial by any means. Recall that comoving distances are computed using the metric, where we set $c = 1$ at the beginning of the course.

- Using the same equation/code as in part 2, we conclude that

$$\theta = 2.402\text{arcseconds}$$

Problem 4 (5 points)

For this question, consider a flat FLRW universe with

$$\begin{aligned} H_0 &= 67.66 \text{ km/s/Mpc} \\ \Omega_\mathrm{m} &= 0.311 \\ \Omega_\mathrm{rad} &= 9.1 \times 10^{-5} \\ \Omega_\Lambda &= 1 - \Omega_\mathrm{m} - \Omega_\mathrm{rad} \end{aligned}$$

a) Plot the age of the Universe as a function of redshift. Remember that $z = 0$ corresponds to the present age of the Universe.

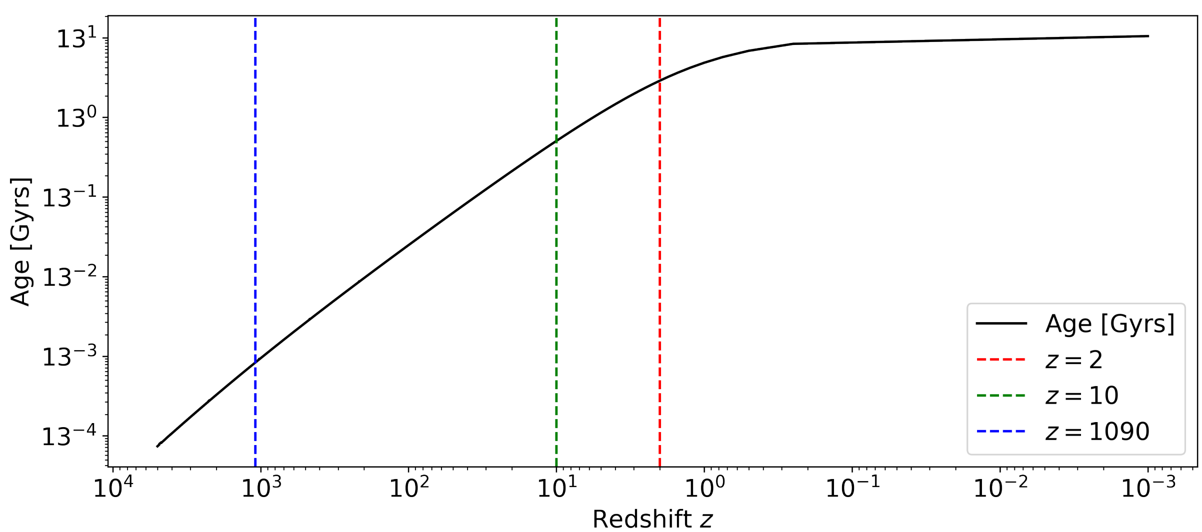
b) Starting at the Big Bang ($z = \infty$), what fraction of the current age of the Universe has elapsed by $z = 2$? What about $z = 10$?

c) How old was the Universe at $z = 1090$, when the photons making up the cosmic microwave background were released?

a) From first principles, the age of the universe is given by the integral

$$\boxed{\text{Using} \quad da = -\frac{dz}{(1+z)^2}} \qquad \begin{aligned} t &= \int dt \\ t(a) &= \int_0^a \frac{da'}{a'H(a')} \\ t(z) &= \int_\infty^z \frac{dz'}{-(1+z')H(z')} \\ &= \int_z^\infty \frac{dz'}{(1+z')H(z')} \end{aligned} \tag{14}$$

We can't plot from $z = 0$ to $z = \infty$, but a sufficiently large value of z suffices.



Find code to reproduce this plot on the next page.

```python
import matplotlib.pyplot as plt
import numpy as np
import scipy.integrate as integrate

# Parameters
Omega_m = 0.311 # Matter
Omega_rad = 9.1e-5 # Radiation
Omega_L = 1-Omega_m-Omega_rad # Cosmological constant.
H0 = 67.66 # km/s/Mpc

km_over_Mpc = 3.086e19
seconds_to_Gyrs = 1/(60*60*24*365*1000000000)

def H(z):
    return H0*np.sqrt(Omega_m*(1+z)**3 + Omega_L + Omega_rad*(1+z)**4)

def integrand(z):
    return 1/((1+z)*H(z))

def age(z):
    age_of_universe, integration_error = integrate.quad(integrand,z,np.inf)
    # The above is in units of [s Mpc / km]. It is good to:
    # 1. Remove units of distance
    # 2. Convert seconds to Gyrs
    age_in_Gyrs = age_of_universe * km_over_Mpc * seconds_to_Gyrs
    return age_in_Gyrs

# The age today is ~ 13 Gyrs. This is a good way to check that the code works:
print(f"Age today: {age(0):.3f} Gyrs")

# Plot age as a function of z
fig, ax = plt.subplots(figsize=(12, 5))
ax.tick_params(axis='both', labelsize=15)
Z = np.linspace(0.001,4999,20000)
plt.plot(Z,[age(ZZ) for ZZ in Z],label="Age [Gyrs]",color="black")
plt.yscale("log", base=13)
plt.xscale("log")
plt.xlabel(r"Redshift $z$",fontsize=15)
plt.ylabel("Age [Gyrs]",fontsize=15)
plt.axvline(x=2,color="red",label=rf"$z=2$",linestyle="--")
plt.axvline(x=10,color="green",label=rf"$z=10$",linestyle="--")
plt.axvline(x=1090,color="blue",label=rf"$z=1090$",linestyle="--")
plt.legend(fontsize=15)
ax.invert_xaxis()
plt.savefig("problem_4_age_as_a_function_of_z.png",dpi=400, transparent=True, bbox_inches='tight')
```

Where the relevant unit conversion factors were added to return age in units of Gyrs.

b) Now that we constructed a python function that returns the age at a given z (`age(z)`), it is a simple matter of calling it at the specified values. We observe that

$$\text{Age}(z = 0) = 13.7980 \tag{15}$$
$$\text{Age}(z = 2) = 3.2793 \tag{16}$$
$$\text{Age}(z = 10) = 0.4716 \tag{17}$$

By $z = 2$, $\frac{3.2793}{13.7980} \cdot 100\% = 23.77\%$ of the current age has elapsed.

By $z = 10$, $\frac{0.4716}{13.7980} \cdot 100\% = 3.42\%$ of the current age has elapsed.

c) At $z = 1090$, the universe was 0.000372 Gyrs old (372 kiloyears).

Problem 5 (10 points)

Observations of Type Ia supernovae were instrumental in the discovery of the acceleration of the expansion and the related presence of dark energy. In this question, we will use some recent supernova data to indeed visually show that a flat universe with $\Omega_\Lambda = 0.7$ and $\Omega_\mathrm{m} = 0.3$ is a much better fit to the data than a matter-dominated universe with $\Omega_\mathrm{m} = 1$.

Start by downloading the data here (a link is also posted on the course webpage). The file contains three columns, giving the redshift (z), the apparent magnitude of the supernova m, and the error on the apparent magnitude Δm, respectively.

a) Using your favorite plotting package, plot m versus z for all supernovae in the data file. Make sure to include the error bar on m for each data point. Clearly label your axes.

b) Now, add the predictions for the two cosmological models mentioned above:

$$\begin{aligned} &\text{Model 1}: \Omega_\Lambda = 0.7,\ \Omega_\mathrm{m} = 0.3 \\ &\text{Model 2}: \Omega_\mathrm{m} = 1 \end{aligned}$$

on your plot. To do so, remember that the predicted apparent magnitude $m_\mathrm{pred}(z)$ for a supernova at redshift z is related to the absolute supernova magnitude M and the distance modulus $\mu(z)$ by

$$m_\mathrm{pred}(z) = \mu(z) + M \tag{18}$$

where

$$M = -19.4 \tag{19}$$

$$\mu(z) = 5\log_{10}\left(\frac{d_L(z)}{\mathrm{Mpc}}\right) + 25 \tag{20}$$

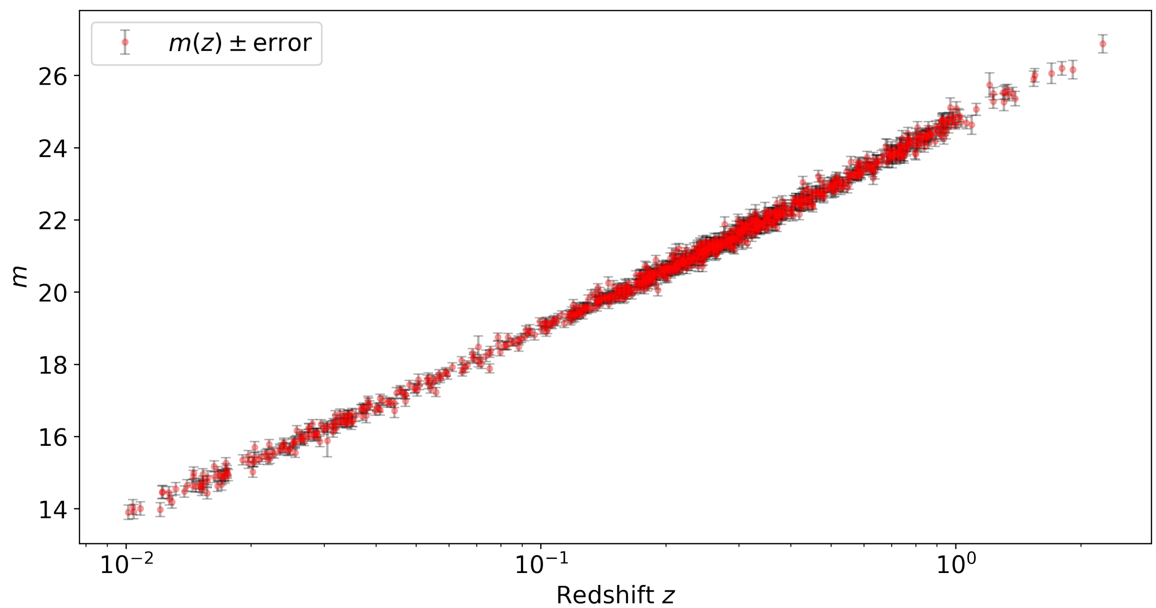
where $d_L(z)$ is the luminosity distance to redshift z in Mpc. Use $H_0 = 67.5$ km/s/Mpc and be mindful of the units when computing d_L. Make sure the two models and the data points (with their error bars) are clearly visible on your plot.

c) Which model appears to be a better fit to the data? Is this conclusion robust to changing M?

a) We can use the following code to plot the data along with the error bars:

```python
import matplotlib.pyplot as plt
import numpy as np

data = np.loadtxt("Pantheon_data.dat").T
# By adding .T, we are transposing the array generated from the .dat file.
# This way, we have
# data[0]   A list of redshifts
# data[1]   A list of m
# data[2]   A list of errors in m

# Plot
fig, ax = plt.subplots(figsize=(12, 6))
```

```python
ax.tick_params(axis='both', labelsize=15)
plt.errorbar(
    data[0], data[1], yerr=data[2],
    fmt='o',
    markersize=3.5,
    color='red',
    alpha=0.3,
    ecolor='black',
    capsize=3,
    label=r"$m(z)\pm$error"
)
plt.xscale("log")
plt.xlabel(r"Redshift $z$",fontsize=15)
plt.ylabel(r"$m$",fontsize=15)
plt.legend(fontsize=15)
plt.savefig("problem_5_data_plot.png",dpi=400, transparent=True,bbox_inches='
    tight')
```



b) Recall that

$$d_L = (z+1)S_k(\chi) \tag{21}$$

Where S_k in a spatially-flat universe follows as

$$\begin{aligned} S_k(\chi) &= \chi \\ &= \int_0^z \frac{dz'}{H(z')} \end{aligned} \tag{22}$$

The Hubble rates for each model is given by:

$$\text{Model 1:} \quad H(z) = H_0\sqrt{\Omega_\Lambda + \Omega_\mathrm{m}(1+z)^3} \tag{23}$$

$$\text{Model 2:} \quad H(z) = H_0\sqrt{\Omega_\mathrm{m}(1+z)^3} \tag{24}$$

The only unit-sensitive function we need to code is H. We need $H(z)$ to be in 1/Mpc. We can introduce a factor of c and a conversion from kilometers to meters to fix this:

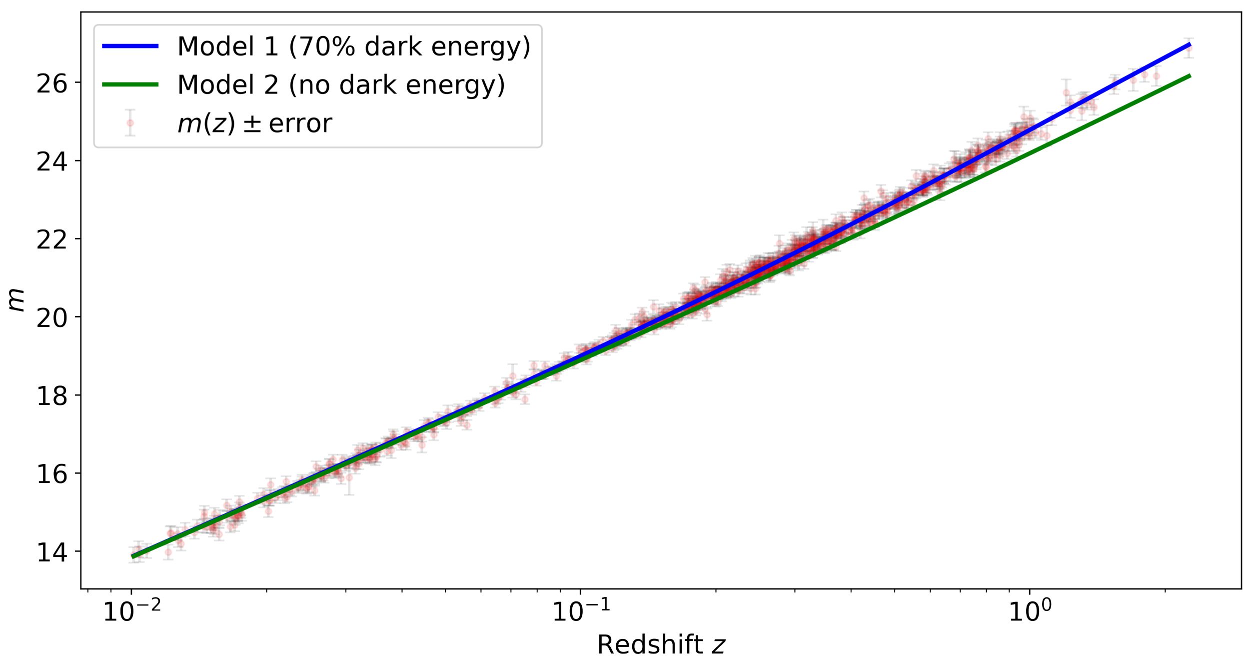
$$H_0 = 67.5\,\frac{\mathrm{km}}{\mathrm{s}\cdot\mathrm{Mpc}} = 67.5\,\frac{\mathrm{km}}{\mathrm{s}\cdot\mathrm{Mpc}}\cdot c^{-1}\frac{\mathrm{s}}{\mathrm{m}}\cdot 1000\frac{\mathrm{m}}{\mathrm{km}} \tag{25}$$

We can use the code

```
import matplotlib.pyplot as plt
import numpy as np
import scipy.integrate as integrate

# Parameters
model_1 = [0.3,0.7] # Omega_m, Omega_L
model_2 = [1.0,0.0] # Omega_m, Omega_L
c = 3e8 # m/s
H0 = 67.5 * 1000 / c # 1/Mpc
M = -19.4
def integrand(z,fractions):
    Omega_m = fractions[0]
    Omega_L = fractions[1]
    return 1/(H0*np.sqrt(Omega_m*(z+1)**3 + Omega_L))

def Sk(z,fractions): #Comoving distance
    com_dist, integration_error = integrate.quad(integrand,0,z,args=fractions)
    return com_dist

def dL(z,fractions):
    return (z + 1)*Sk(z,fractions)

def mu(z,fractions):
    return 5*np.log10(dL(z,fractions)) + 25

def m_pred(z,fractions):
    return mu(z,fractions) + M

data = np.loadtxt("Pantheon_data.dat").T

# Plot of data
fig, ax = plt.subplots(figsize=(12, 6))
ax.tick_params(axis='both', labelsize=15)
plt.errorbar(
    data[0], data[1], yerr=data[2],
    fmt='o',
    markersize=3.5,
    color='red',
    alpha=0.1,
    ecolor='black',
    capsize=3,
    label=r"$m(z)\pm$error",
    zorder=0
)
plt.xscale("log")
plt.xlabel(r"Redshift $z$",fontsize=15)
plt.ylabel(r"$m$",fontsize=15)

```

```

# Compute predictions

Z = np.linspace(np.min(data[0]),
                np.max(data[0]),
                500)

plt.plot(Z,
         [m_pred(ZZ,model_1) for ZZ in Z],
         label="Model 1 (70% dark energy)",
         zorder=2,
         color="blue",
         linewidth=2.5)

plt.plot(Z,
         [m_pred(ZZ,model_2) for ZZ in Z],
         label="Model 2 (no dark energy)",
         zorder=2,
         color="green",
         linewidth=2.5)

plt.legend(fontsize=15)
plt.savefig("problem_5_data_with_predictions.png",dpi=400, transparent=True,
    bbox_inches='tight')
```

To generate the plot



c) The predictions from Model 1 (with dark energy) are much closer to the observed data.

Problem 6 (4 points)

For this question, assume the same cosmological parameters given in question 4 above, unless otherwise noted.

a) Compute the redshift at which the energy density in dark energy is equal to that of matter. How much time has elapsed betwen that epoch and today? Compare this timescale to the age of the Solar system (4.6 Gyrs). This is often referred to as the "coincidence" problem.

b) Imagine a universe in which there are 2 extra species of massless neutrinos (in addition to those present in the Standard Model of particle physics). What is the redshift of matter-radiation equality in such a universe? Assume that the extra neutrinos have the same temperature as the standard ones.

a) From problem 4, recall that

$$\begin{aligned} H_0 &= 67.66 \text{ km/s/Mpc} \\ \Omega_\text{m} &= 0.311 \\ \Omega_\Lambda &= 0.688909 \end{aligned} \tag{26}$$

The density parameters scale as:

$$\Omega_\text{m}(z) = \left(\frac{H_0}{H(z)}\right)^2 (z+1)^3 \Omega_{\text{m},0} \tag{27}$$

$$\Omega_\Lambda(z) = \left(\frac{H_0}{H(z)}\right)^2 \Omega_{\Lambda,0} \tag{28}$$

If we set Eqs. 27 and 28 equal to each other, we see that

$$\begin{aligned} (z_\text{eq}+1)^3 &= \frac{\Omega_{\Lambda,0}}{\Omega_{\text{m},0}} \\ z_\text{eq} &= \sqrt[3]{\frac{\Omega_{\Lambda,0}}{\Omega_{\text{m},0}}} - 1 \\ &= 0.304 \end{aligned} \tag{29}$$

Using the python functions defined on problem 4, we observe that

$$\text{Age}(z = 0.304) = 10.229 \text{ Gyrs} \tag{30}$$

In other words, the density parameters were equal

$$13.798 - 10.229 = 3.569 \text{ Gyrs} \tag{31}$$

Which is in the same timescale as the time of the Solar system's creation.

b) Computing the matter-radiation equality follows as above (with $\Omega_{\rm rad}$ instead of Ω_Λ), but we must first find the right $\Omega_{\rm rad,0}$, since the one given in problem 4

$$\Omega_{\rm rad,0} = 9.1 \times 10^{-5} \qquad \text{(From problem 4)}$$

Only includes the standard model (SM) neutrinos. Once we find the neutrino (SM+BeyondSM neutrinos) density parameter, as well as the photon contribution (recall that we are only given the total radiation budget of the universe), we can get the right value for $\Omega_{\rm rad,0}$. From the `Energy Content of the Universe` lecture notes, recall that

$$\begin{aligned}\Omega_{\gamma,0} &= 2.47 \times 10^{-5} h^{-2} \\ &= 5.395 \times 10^{-5}\end{aligned} \tag{32}$$

This number does not change here. Further, $h = 67.66/100$ in this case. For neutrinos:

$$\begin{aligned}\Omega_{\nu,0} &= 1.68 \times 10^{-5} h^{-2} \left(\frac{N_{\rm eff}}{3}\right) \\ &= \begin{cases} 3.723 \times 10^{-5} & \text{if } N_{\rm eff} = 3.044 \\ 6.17 \times 10^{-5} & \text{if } N_{\rm eff} = 2 + 3.044 \end{cases}\end{aligned} \tag{33}$$

The first case was the one used in problem 4 (hence $\Omega_{\rm rad,0} = 9.1 \times 10^{-5}$ in problem 4). Using the second case ($N_{\rm eff} = 5.044$) we get:

$$\Omega_{\rm rad,0} = 1.156 \times 10^{-4} \tag{34}$$

Equality happens when $\Omega_{\rm m} = \Omega_{\rm rad}$, where

$$\Omega_{\rm rad}(z) = \left(\frac{H_0}{H(z)}\right)^2 (z+1)^4 \Omega_{\rm rad,0} \tag{35}$$

So

$$\begin{aligned}(z_{\rm eq}+1)^4 \Omega_{\rm rad,0} &= (z_{\rm eq}+1)^3 \Omega_{\rm m,0} \\ z_{\rm eq} + 1 &= \frac{\Omega_{\rm m,0}}{\Omega_{\rm rad,0}} \\ z_{\rm eq} &= \frac{\Omega_{\rm m,0}}{\Omega_{\rm rad,0}} - 1 \\ &= \frac{0.311}{1.156 \times 10^{-4}} - 1 \\ &= 2689.31\end{aligned} \tag{36}$$

A number that's smaller (in other words, later in time) compared to the standard model time (redshift) corresponding to matter-radiation equality: $z_{{\rm eq},SM} \simeq 3400$.