ASTR 425/525
Homework 3 solutions

Fall 2025

Due October 22nd

Contents

Problem 1 (8 points)

Consider the exact expression for the number density of a fermion of mass m in thermal equilibrium with zero chemical potential

$$n = g \int \frac{d^3p}{(2\pi)^3} \frac{1}{\exp\left(\sqrt{p^2+m^2}/T\right)+1} \tag{1}$$

a) Using $x = p/T$ and $y = m/T$, show that the above can be rewritten in the following form:

$$\frac{n(y)}{T^3} = \frac{g}{2\pi^2} \int_0^\infty dx \frac{x^2}{e^{\sqrt{x^2+y^2}}+1} \tag{2}$$

b) Using numerical integration and a plotting software of your choice, plot $n(y)/T^3$ as a function of $y = m/T$. Assume that $g = 2$ (like for an electron). Clearly label your axes.

c) Now, add to your plot both the relativistic ($y \ll 1$) and non-relativistic ($y \gg 1$) limit for the number density of this fermion that we derived in class. Remember to convert them to functions of y, and plot $n(y)/T^3$ for those too. Confirm that these results match the exact result plotted in part (b) in the appropriate limit.

a) Consider rewriting the expression as

$$n = g \int \frac{d^3p}{(2\pi)^3} \frac{1}{\exp\left(\sqrt{\left(\frac{p}{T}\right)^2+\left(\frac{m}{T}\right)^2}\right)+1} \tag{3}$$

Recalling that $d^3p \equiv 4\pi p^2 dp$ (integrating the angular part out), we see that

$$\begin{aligned} n &= g\frac{4\pi}{(2\pi)^3} \int_0^\infty \frac{p^2 dp}{\exp\left(\sqrt{\left(\frac{p}{T}\right)^2+\left(\frac{m}{T}\right)^2}\right)+1} \\ &= g\frac{1}{2\pi^2} \int_0^\infty \frac{p^2 dp}{\exp\left(\sqrt{\left(\frac{p}{T}\right)^2+\left(\frac{m}{T}\right)^2}\right)+1} \end{aligned} \tag{4}$$

Since the variable $y = m/T$ is independent of the integration variables, its introduction can be inserted directly:

$$n = g\frac{1}{2\pi^2} \int_0^\infty dp \frac{p^2 dp}{\exp\left(\sqrt{\left(\frac{p}{T}\right)^2+y^2}\right)+1} \tag{5}$$

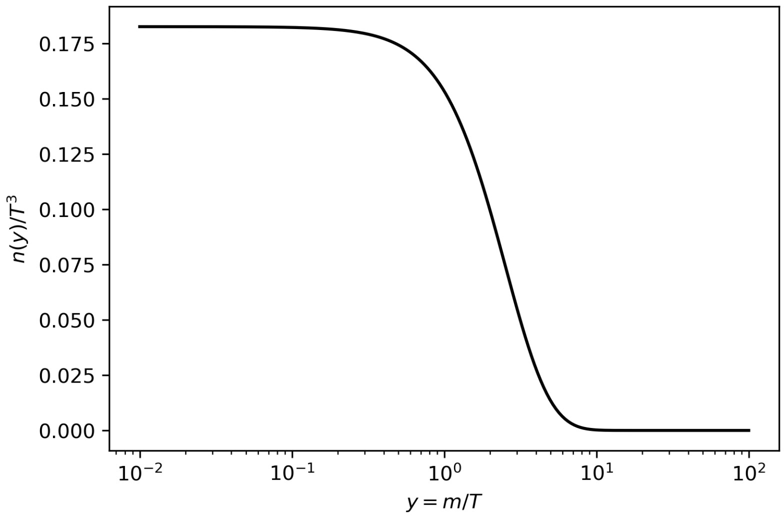
To introduce x, we note that $dp = T dx$, so

$$\begin{aligned} n &= g\frac{1}{2\pi^2} \int_0^\infty dp \frac{(xT)^2 T dx}{\exp\left(\sqrt{x^2+y^2}\right)+1} \\ &= g\frac{1}{2\pi^2} T^3 \int_0^\infty dp \frac{x^2 dx}{\exp\left(\sqrt{x^2+y^2}\right)+1} \end{aligned} \tag{6}$$

From which we can conclude that

$$\frac{n(y)}{T^3} = \frac{g}{2\pi^2}\int_0^\infty dx \frac{x^2}{e^{\sqrt{x^2+y^2}}+1} \tag{7}$$

b) We can use `scipy.integrate.quad` for this:




```python
import scipy.integrate as integrate
import numpy as np
import matplotlib.pyplot as plt

def integrand(x,y):
    return x**2 /(np.exp(np.sqrt(x**2 + y**2))+1)

def n(y,g=2): # over T^3
    pre_factor = g/(2*np.pi**2)
    integral, error = integrate.quad(lambda x: integrand(x,y), 0, np.inf)
    return pre_factor * integral

y = np.logspace(-2,2,200)

n_over_T_cubed = [n(yy,2) for yy in y]

plt.plot(y,n_over_T_cubed,color="black")
plt.xscale("log")
plt.xlabel(r"$y=m/T$")
plt.ylabel(r"$n(y)/T^3$")
plt.savefig("p1_part_b.png",dpi=300,bbox_inches='tight')
```

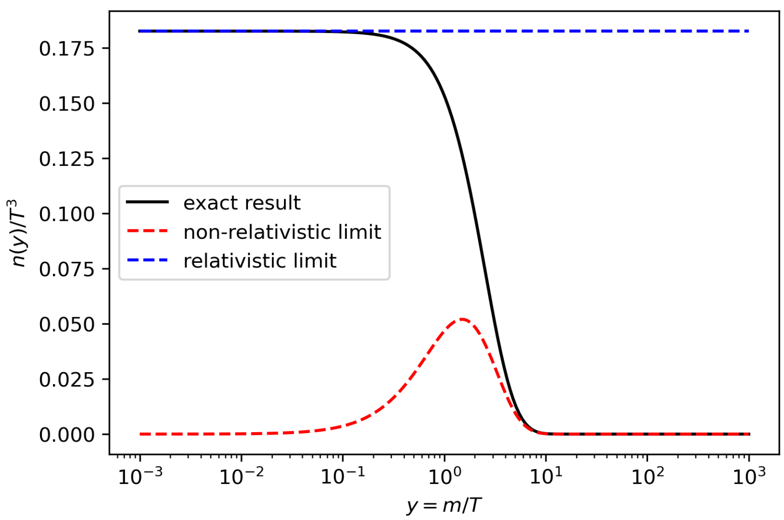
c) Starting from the expression for $n_{\rm NR}$:

$$
\begin{aligned}
n_{\rm NR} &= g\left(\frac{mT}{2\pi}\right)^{3/2} e^{-m/T} \\
\frac{n_{\rm NR}}{T^3} &= g\left(\frac{m}{2\pi}\right)^{3/2} \frac{T^{3/2}}{T^3} e^{-y} \\
&= g\left(\frac{m}{2\pi}\right)^{3/2} T^{-3/2} e^{-y} \\
&= g\left(\frac{m}{2\pi T}\right)^{3/2} e^{-y} \\
&= g\left(\frac{y}{2\pi}\right)^{3/2} e^{-y}
\end{aligned}
\tag{8}
$$

For the relativistic limit, observe that $n_{\rm Rel}/T^3$ is a constant:

$$
\frac{n_{\text{Rel, fermions}}}{T^3} = \frac{3}{4}\frac{g\zeta(3)}{\pi^2} \tag{9}
$$

The plot follows:



Based on the code for part (b):

```python
import scipy.integrate as integrate
import numpy as np
import matplotlib.pyplot as plt

def integrand(x,y):
    return x**2 /(np.exp(np.sqrt(x**2 + y**2))+1)

def n(y,g=2): # over T^3
    pre_factor = g/(2*np.pi**2)
    integral, error = integrate.quad(lambda x: integrand(x,y), 0, np.inf)
    return pre_factor * integral

def n_non_rel(y,g=2): # over T^3
```

```
    return g * np.exp(-y)*(y/(2*np.pi))**(3/2)

zeta_3 = 1.20206

def n_rel(g=2): # over T^3
    # This is constant
    return g * (3/4) * zeta_3 / (np.pi**2)

y = np.logspace(-3,3,200)

n_over_T_cubed = [n(yy,2) for yy in y]

n_over_T_cubed_non_rel = [n_non_rel(yy,2) for yy in y]

n_over_T_cubed_rel = [n_rel(2) for yy in y]

plt.plot(y,n_over_T_cubed,color="black",label="exact result")
plt.plot(y,n_over_T_cubed_non_rel,color="red",linestyle="--",label="non-relativistic limit")
plt.plot(y,n_over_T_cubed_rel,color="blue",linestyle="--",label="relativistic limit")
plt.xscale("log")
plt.xlabel(r"$y=m/T$")
plt.ylabel(r"$n(y)/T^3$")
plt.legend(loc="center left")
plt.savefig("p1_part_c.png",dpi=300,bbox_inches='tight')
```

Problem 2 (6 points)

When the chemical potential μ of a given species vanishes, the number of particles (n) and antiparticles ($\overline{n}$) of that species in the cosmic plasma are equal. However, the presence of $\mu \neq 0$ results in a net particle number density, $n - \overline{n} \neq 0$.

a) For relativistic fermions with $\mu \neq 0$ and $T \gg m$, show that

$$\begin{aligned} n - \overline{n} &= \frac{g}{2\pi^2} \int_0^\infty dp\, p^2 \left(\frac{1}{\exp\left(\frac{p-\mu}{T}\right) + 1} - \frac{1}{\exp\left(\frac{p+\mu}{T}\right) + 1} \right) \\ &= \frac{g}{6\pi^2} T^3 \left[\pi^2 \left(\frac{\mu}{T}\right) + \left(\frac{\mu}{T}\right)^3 \right] \end{aligned} \tag{10}$$

Note that this is an exact result in the relativistic limit and not a truncated series.

b) Now, let's apply the above result to electrons and positrons in their relativistic limit ($T \gg 1$ MeV). Electric charge neutrality in the early universe implies that $n_p = n_e - \overline{n}_e$, where n_p is the number density of protons. Use this to show that, in the early universe, we have

$$\frac{\mu_e}{T} \simeq \frac{3\eta_b \zeta(3)}{\pi^2}$$

where $\eta_b \equiv n_b/n_\gamma$ is the baryon-to-photon ratio, with n_b and n_γ being the baryon and photon number densities, respectively.

Hint: Relate the proton number density to the baryon number density. Do not forget the presence of neutrons. What is the relative abundance of neutrons and protons for $T \gg 1$ MeV?

a) Recall that (exactly)

$$n = \frac{g}{2\pi^2} \int_0^\infty dp\, \frac{p^2}{\exp\left(\left[E - \mu\right]/T\right) + 1} \tag{11}$$

Now, in the relativistic limit, we have

$$\begin{aligned} & p \gg m \\ \Rightarrow \quad & E \simeq p \end{aligned} \tag{12}$$

So

$$n = \frac{g}{2\pi^2} \int_0^\infty \frac{p^2}{\exp\left(\left[p - \mu\right]/T\right) + 1} \tag{13}$$

Similarly for $\overline{n}$:

$$\overline{n} = \frac{g}{2\pi^2} \int_0^\infty \frac{p^2}{\exp\left(\left[p + \mu\right]/T\right) + 1} \tag{14}$$

Together it is then clear that

$$n - \overline{n} = \frac{g}{2\pi^2} \int_0^\infty dp\, p^2 \left(\frac{1}{\exp\left(\frac{p-\mu}{T}\right) + 1} - \frac{1}{\exp\left(\frac{p+\mu}{T}\right) + 1} \right) \tag{15}$$

If we introduce the variables

$$\begin{aligned} x &= p/T \\ y &= \mu/T \end{aligned} \tag{16}$$

Then $p = xT$ and $dp = Tdx$, so that

$$\begin{aligned} n &= \frac{g}{2\pi^2}\int_0^\infty \frac{(Tx)^2 T dx}{\exp(x-y)+1} \\ &= \frac{gT^3}{2\pi^2}\int_0^\infty \frac{x^2 dx}{\exp(x-y)+1} \end{aligned} \tag{17}$$

Similarly for $\overline{n}$:

$$\overline{n} = \frac{gT^3}{2\pi^2}\int_0^\infty \frac{x^2 dx}{\exp(x+y)+1} \tag{18}$$

We can now introduce a 3rd substitution (different for each integral) in the form of a shift:

$$\begin{aligned} &(\text{for } n) \quad u = x - y \\ &(\text{for } \bar{n}) \quad u = x + y \end{aligned} \tag{19}$$

So that

$$\begin{aligned} (n-\overline{n})\frac{2\pi^2}{gT^3} &= \int_0^\infty \frac{x^2}{e^{x-y}+1} - \frac{x^2}{e^{x+y}+1}\, dx \\ &= \int_{-y}^\infty \frac{(u+y)^2 du}{e^u+1} \; - \; \int_y^\infty \frac{(u-y)^2 du}{e^u+1} \\ &= \int_{-y}^y \frac{(u+y)^2 du}{e^u+1} \; + \; \int_y^\infty \frac{(u+y)^2 du}{e^u+1} - \int_y^\infty \frac{(u-y)^2 du}{e^u+1} \\ &= \int_{-y}^y \frac{(u+y)^2 du}{e^u+1} \; + \; \int_y^\infty \frac{(u+y)^2 - (u-y)^2 \; du}{e^u+1} \\ &= \int_{-y}^y \frac{u^2+2uy+y^2}{e^u+1} du \; + \; \int_y^\infty \frac{4yu}{e^u+1} du \end{aligned} \tag{20}$$

Let's focus on the first term. Observing that

$$\frac{1}{e^u+1} + \frac{1}{e^{-u}+1} = 1 \tag{21}$$

We can split the $[-y, y]$ integral into two, one in $[-y, 0]$ and one in $[0, y]$ and manipulate them using the identity above:

$$
\begin{aligned}
\int_{-y}^{y} \frac{u^2+2uy+y^2}{e^u+1}du &= \int_{-y}^{0} \frac{u^2+2uy+y^2}{e^u+1}du \quad + \quad \int_{0}^{y} \frac{u^2+2uy+y^2}{e^u+1}du \\
&= \int_{0}^{y} \frac{u^2-2uy+y^2}{e^{-u}+1}du \quad + \quad \int_{0}^{y} \frac{u^2+2uy+y^2-2uy+2uy}{e^u+1}du \\
&= \int_{0}^{y} \frac{u^2-2uy+y^2}{e^{-u}+1}du \quad + \quad \int_{0}^{y} \frac{u^2-2uy+y^2+4uy}{e^u+1}du \\
&= \int_{0}^{y} \frac{u^2-2uy+y^2}{e^{-u}+1}du \quad + \quad \int_{0}^{y} \frac{u^2-2uy+y^2}{e^u+1}du \quad + \quad \int_{0}^{y} \frac{4uy}{e^u+1}du \qquad (22) \\
&= \int_{0}^{y} \left(u^2-2uy+y^2\right)\left(\cancel{\frac{1}{e^u+1}+\frac{1}{e^{-u}+1}}\right)du \quad + \quad \int_{0}^{y} \frac{4uy}{e^u+1}du \\
&= \int_{0}^{y} \left(u^2-2uy+y^2\right)du \quad + \quad \int_{0}^{y} \frac{4uy}{e^u+1}du \\
&= \frac{y^3}{3} \quad + \quad \int_{0}^{y} \frac{4uy}{e^u+1}du
\end{aligned}
$$

Up to this point, we have

$$
(n-\overline{n})\frac{2\pi^2}{gT^3} = \frac{y^3}{3} \quad + \quad \int_{0}^{y} \frac{4uy}{e^u+1}du \quad + \quad \int_{y}^{\infty} \frac{4yu}{e^u+1} \tag{23}
$$

We can merge the remaining two integrals into one in $[0, \infty]$, where we can make use of the identity:

$$
\int_{0}^{\infty} dx \frac{x}{e^x+1} = \frac{\pi^2}{12} \tag{24}
$$

It follows that

$$
\begin{aligned}
(n-\overline{n})\frac{2\pi^2}{gT^3} &= \frac{y^3}{3} \quad + \quad \int_{0}^{y} \frac{4uy}{e^u+1}du \quad + \quad \int_{y}^{\infty} \frac{4yu}{e^u+1} \\
&= \frac{y^3}{3} \quad + \quad 4y\int_{0}^{\infty} \frac{u}{e^u+1}du \\
&= \frac{y^3}{3} \quad + \quad 4y\frac{\pi^2}{12} \\
&= \frac{y^3}{3} \quad + \quad \frac{\pi}{3}y \qquad (25) \\
\text{(plugging back } y=\mu/T\text{)} \qquad &= \frac{1}{3}\left(\frac{\mu}{T}\right)^3 + \frac{\pi}{3}\frac{\mu}{T} \\
(n-\overline{n}) &= \frac{g}{2\pi^2}T^3\left[\frac{1}{3}\left(\frac{\mu}{T}\right)^3 + \frac{\pi}{3}\frac{\mu}{T}\right] \\
&= \frac{g}{6\pi^2}T^3\left[\pi^2\left(\frac{\mu}{T}\right) + \left(\frac{\mu}{T}\right)^3\right]
\end{aligned}
$$

as expected.

Note: If you use an integral calculator (such as `Integrate[]` in Mathematica). You will get PolyLogs, in fact:

$$\begin{aligned} n - \overline{n} &= \frac{g}{2\pi^2}\Bigg(- 2T^2 \mathrm{Li}_3\left[-e^{\mu/T}\right] - -2T^3 \mathrm{Li}_3\left[-e^{-\mu/T}\right]\Bigg) \\ &= \frac{g}{2\pi^2}2T^3\Bigg(\mathrm{Li}_3\left[(-e^{\mu/T})^{-1}\right] - \mathrm{Li}_3\left[-e^{\mu/T}\right]\Bigg) \end{aligned} \tag{26}$$

These PolyLogs are **tri**logarithms ($\mathrm{Li}_3[\cdot]$), and they satisfy the following identity:

$$\mathrm{Li}_3\left[x^{-1}\right] - \mathrm{Li}_3[x] = \frac{1}{6}\ln^3(-x) + \frac{\pi^2}{6}\ln(-x) \tag{27}$$

Where in our case, $x = -e^{\mu/T}$, so

$$\begin{aligned} n - \overline{n} &= \frac{g}{2\pi^2}2T^3\left(\frac{1}{6}\left[\frac{\mu}{T}\right]^3 + \frac{\pi^2}{6}\frac{\mu}{T}\right) \\ &= \frac{g}{6\pi^2}T^3\left[\pi^2\left(\frac{\mu}{T}\right) + \left(\frac{\mu}{T}\right)^3\right] \end{aligned} \tag{28}$$

as before.

b) At high temperatures (above 1 MeV), neutrons and protons were in thermal equilibrium, in such a way that their chemical potentials were the same:

$$\begin{aligned} p + \overline{\nu}_{\mathrm{e}} &\leftrightarrow n + e^+ \\ p + e^- &\leftrightarrow n + \nu_{\mathrm{e}} \\ \mu_{\mathrm{n}} &= \mu_{\mathrm{p}} \end{aligned} \tag{29}$$

So $n_{\mathrm{p}} \sim n_{\mathrm{n}}$, which in turn implies

$$n_{\mathrm{b}} \sim 2n_{\mathrm{p}} \tag{30}$$

Further, at very high temperatures we have

$$\begin{aligned} n - \overline{n} &\simeq \frac{g}{6\pi^2}T^3\pi^2\left(\frac{\mu}{T}\right) \\ &= \frac{g}{6}T^2\mu_{\mathrm{e}} \end{aligned} \tag{31}$$

With this in mind, we see that

$$\begin{aligned} n_{\mathrm{p}} &= n_{\mathrm{e}} - \overline{n}_{\mathrm{e}} \\ \frac{1}{2}n_{\mathrm{b}} &= \frac{g}{6}T^2\mu_{\mathrm{e}} \\ n_{\mathrm{b}} &= \frac{g}{3}T^2\mu_{\mathrm{e}} \end{aligned} \tag{32}$$

Now, let's introduce η_{b} (the baryon photon ratio)

$$\eta_{\mathrm{b}} \equiv \frac{n_{\mathrm{b}}}{n_\gamma} \tag{33}$$

in order to eliminate $n_{\rm b}$. Recalling that

$$\text{(for relatisivtic bosons)} \quad n(T) = \frac{g\zeta(3)}{\pi^2}T^3 \tag{34}$$

we see that

$$\begin{aligned} n_{\rm b} &= \frac{g}{3}T^2\mu_{\rm e} \\ n_\gamma \eta_{\rm b} &= \frac{g}{3}T^2\mu_{\rm e} \\ \frac{g\zeta(3)}{\pi^2}T^3\eta_{\rm b} &= \frac{g}{3}T^2\mu_{\rm e} \\ \frac{\mu_{\rm e}}{T} &= \frac{g\zeta(3)}{\pi^2}\cdot\eta_{\rm b}\frac{3}{g} \\ &= \frac{3\eta_{\rm b}\zeta(3)}{\pi^2} \end{aligned} \tag{35}$$

As expected.

Problem 3 (6 points)

A we discussed in class, cosmological neutrinos are slightly colder than cosmic microwave background photons today with

$$T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma \tag{36}$$

where T_ν is the neutrino temperature, and T_γ is the photon temperature.

a) Show that the combined number density of one generation of neutrinos and anti-neutrinos in the Universe today is

$$n_\nu = \frac{3}{11} n_\gamma, \tag{37}$$

where n_γ is the number density of photons today.

b) Use the result above to show that the contribution from all species of non-relativistic massive neutrinos to the critical density of the Universe today is given by

$$\Omega_\nu h^2 = \frac{\Sigma_i m_{\nu,i}}{94\text{ eV}} \tag{38}$$

where h is the reduced Hubble rate, and $\sum_i m_{\nu,i}$ denotes the sum of neutrinos masses, summed over all neutrino species.

a) Recall that in the relativistic limit, for bosons (photons)

$$n_\gamma = g_\gamma \frac{\zeta(3)}{\pi^2} T_\gamma^3 \tag{39}$$

And for fermions (neutrinos):

$$n_\nu = \frac{3}{4} g_\nu \frac{\zeta(3)}{\pi^2} T^3 \tag{40}$$

Now, let's use the fact that $g_\nu = g_\gamma$ (in this scenario of 1 generation) and the fact that $T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma$ to see that

$$\begin{aligned} n_\nu &= \frac{3}{4} g_\nu \frac{\zeta(3)}{\pi^2} T^3 \\ &= \frac{3}{\cancel{4}} g_\gamma \frac{\zeta(3)}{\pi^2} \frac{\cancel{4}}{11} T_\gamma^3 \\ &= \frac{3}{11} \cdot \frac{\zeta(3) g_\gamma}{\pi^2} T_\gamma \\ &= \frac{3}{11} \cdot n_\gamma \end{aligned} \tag{41}$$

b) Let's be a bit mindful here: In part (a), we considered a relativistic scenario. We now want the non-relativistic counterpart. Luckily, the number density is conserved, so we can still use $n_\nu = \frac{3}{11} n_\gamma$. Now, non-relativistic physics shows up when we bring in the fact that

$$\rho_i \simeq m_i n_i \tag{42}$$

For neutrinos, and considering 3 species, we have:

$$\begin{aligned}\rho_\nu &= \sum_{i\in\{\text{flavors}\}} m_{\nu,i} n_{\nu,i} \\ &= \left(\sum_i m_{\nu,i}\right) n_\nu \\ &= \frac{3}{11}\left(\sum_i m_{\nu,i}\right) n_\gamma\end{aligned} \tag{43}$$

We can now compute

$$\begin{aligned}\Omega_\nu h^2 &= \frac{\rho_\nu}{\rho_\mathrm{c}} h^2 \\ &= \frac{h^2}{\rho_\mathrm{c}}\frac{3}{11} n_\gamma \left(\sum_i m_{\nu,i}\right)\end{aligned}$$

Recalling that (both values can be found in the `Thermo for relativistic particles` lecture notes)

$$\begin{aligned}n_\gamma(\text{today}) &= 410\ \mathrm{cm}^{-3} \\ \rho_c = \frac{3H_0^2}{8\pi G} &= h^2 \cdot 8.098 \times 10^{-11}\ \mathrm{eV}^4 \\ &= h^2 \cdot 1.0537 \times 10^4\ \mathrm{eV}\cdot\mathrm{cm}^{-3}\end{aligned} \tag{44}$$

So

$$\begin{aligned}\Omega_\nu h^2 &= \frac{h^2}{\rho_\mathrm{c}}\frac{3}{11} n_\gamma \left(\sum_i m_{\nu,i}\right) \\ &= \frac{3}{11}\cdot\frac{h^2}{h^2\cdot 1.0537\times 10^4\ \mathrm{eV}\cdot\mathrm{cm}^{-3}}\cdot(410\ \mathrm{cm}^{-3})\cdot\left(\sum_i m_{\nu,i}\right) \\ &= \frac{1}{94.233\ \mathrm{eV}}\left(\sum_i m_{\nu,i}\right)\end{aligned}$$

As expected.

Problem 4 (8 points)

If the neutrinos were massless in our Universe, the relative energy density of neutrinos as compared to that of photons would be constant for $T \lesssim 0.1$ MeV (after e^+e^- annihilation), that is, $\rho_\nu/\rho_\gamma =$ const.

a) Assuming that all three neutrino species present in our Universe are massless, compute the ratio ρ_ν/ρ_γ for $T \lesssim 0.1$ MeV. Remember that $T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma$.

b) In the real universe, we know that at least two neutrino species have non-vanishing masses. A realistic model compatible with the results of neutrino oscillation experiments is to have one massless neutrino ($m_1 = 0$), a second one with mass $m_2 = 0.01$ eV, and a third one with mass $m_3 = 0.05$ eV.

Compute the ration $\rho_\nu(z)/\rho_\gamma(z)$ in this scenario over the redshift range $0 \leq z \leq 10^3$ and plot it using your favorite plotting package. Clearly label your axes. Add the constant line you found in part (a) to your plot as a comparison. Note that since massive neutrinos always decouple from the cosmic plasma while ultra-relativistic, their energy density (for one species) is given by

$$\rho_\nu(T_\nu) = g \int \frac{d^3p}{(2\pi)^3} \frac{\sqrt{p^2 + m_\nu^2}}{e^{p/T_\nu} + 1} \tag{45}$$

a) For relativistic bosons (photons) we have

$$\rho_\gamma = g_\gamma \frac{\pi^2}{30} T_\gamma^4 \tag{46}$$

For relativistic fermions (massless neutrinos in this case) we have

$$\begin{aligned}\rho_\nu &= g_\nu \frac{7}{8}\frac{\pi^2}{30} T_\nu^4 \\ &= g_\nu \frac{7}{8}\frac{\pi^2}{30} \left(\frac{4}{11}\right)^{4/3} T_\gamma^4\end{aligned} \tag{47}$$

So

$$\begin{aligned}\frac{\rho_\nu}{\rho_\gamma} &= \frac{g_\nu \frac{7}{8}\frac{\pi^2}{30}\left(\frac{4}{11}\right)^{4/3} T_\gamma^4}{g_\gamma \frac{\pi^2}{30} T_\gamma^4} \\ &= \frac{21}{8} \cdot \left(\frac{4}{11}\right)^{4/3}\end{aligned} \tag{48}$$

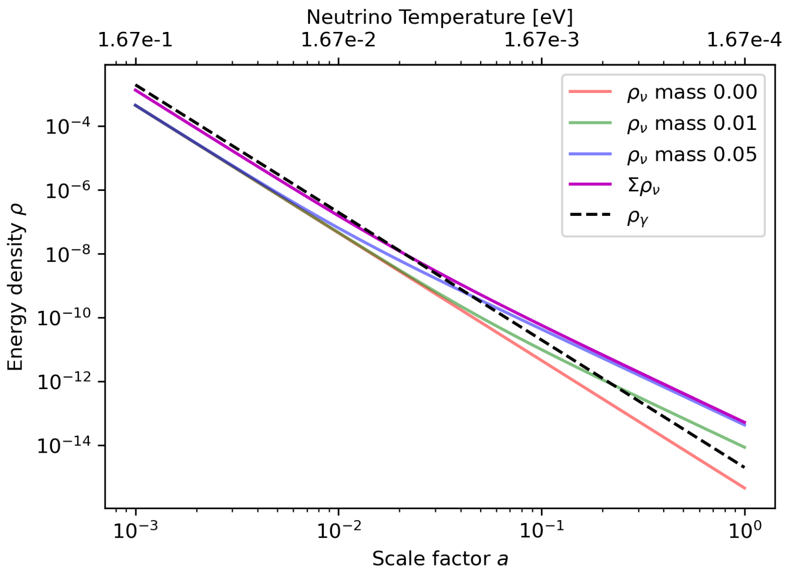
b) We already know $\rho_\gamma(T_\gamma)$. To write it as a function of redshift, recall that $T \propto \frac{1}{a}$, so

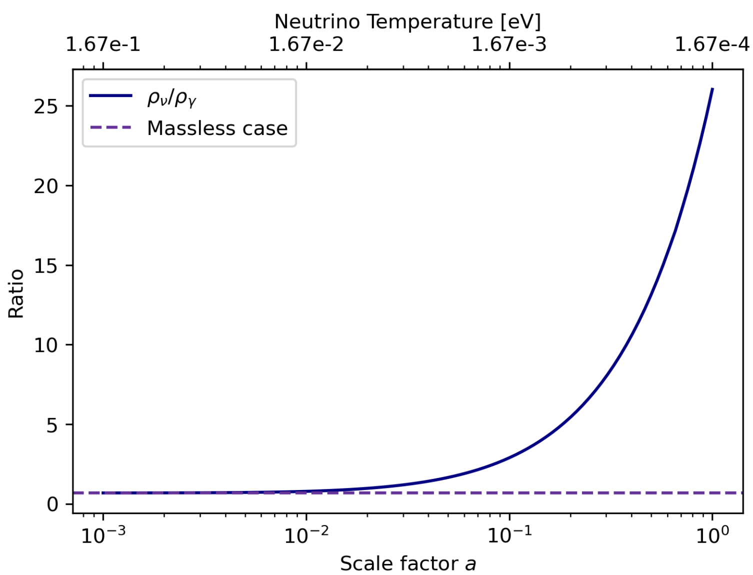
$$T_\gamma = \frac{T_0}{a} = T_0(1+z) \tag{49}$$

Where T_0 is the temperature from the CMB today. For each one of the neutrino flavors, we have

$$\rho_{\nu,i}(T_\nu) = \frac{\cancel{g_{\nu,i}}}{\cancel{2}\pi^2} \int_0^\infty dp \frac{p^2\sqrt{p^2 + m_{\nu,i}^2}}{e^{p/T_\nu} + 1} \tag{50}$$

Where $g_{\nu,i} = 2$ for each of the flavors (hence the cancellation). We can use python to compute the ratio $\frac{\rho_\nu}{\rho_\gamma}$ at all times exactly. (This python code will print out a plot of the energy densities and another plot with the ratio)





For this problem, it is recommended to use integration libraries with more precision, for example `npmath`.

```
import numpy as np
import matplotlib.pyplot as plt
import mpmath as mp
mp.dps = 50

def photon_integrand(p,T):
    return p**2 * p / (mp.exp(p/T)-1)

def neutrino_integrand(p,m,T):
```

```
    return p**2 * mp.sqrt(p**2 + m**2) / (mp.exp(p/T)+1)


def rho_neutrino_flavor(mass,T_photon):
    T_neutrino = (4/11)**(1/3) * T_photon
    integral = mp.quad(
        lambda x: neutrino_integrand(x,mass,T_neutrino),
        [0, mp.inf]
        )
    # g neutrino is 2 per flavor
    return integral / mp.pi**2

def rho_photons(T_photon):
    integral = mp.quad(
        lambda x: photon_integrand(x,T_photon),
        [0, mp.inf]
        )
    # g neutrino is 2 per flavor
    return integral / mp.pi**2

masses = [0.0,0.01,0.05] #eV

# Scale factor values
# We want from today z=0 to z=10^3
scale_factors = np.logspace(-3,0,100)

# CMB temp
T0 = 2.348*10**(-4) # eV

temperatures = T0/scale_factors # photon temperatures

rho_g = np.array([rho_photons(temp) for temp in temperatures])
rho_n = np.array([[rho_neutrino_flavor(m,temp) for temp in temperatures] for
    m in masses])

"""
Double axes stuff
"""

plot_scale_factors = [10**-3,10**-2,10**-1,10**0]
plot_temperatures_neutrino = ["1.67e-1","1.67e-2","1.67e-3","1.67e-4"]

"""
Show individual densities
"""
fig, ax = plt.subplots()
colors = ["red","green","blue"]
for color, mass, rho_one_flavor in zip(colors,masses,rho_n):
    ax.plot(scale_factors,rho_one_flavor,color=color,label=fr"$\rho_\nu$ mass
    {mass:.2f}",alpha=0.5)

ax.plot(scale_factors,sum(rho_n),color="m",label=r"$\Sigma \rho_\nu$ ")

ax.plot(scale_factors,rho_g,color="black",label=r"$\rho_{\gamma }$",linestyle
    ="--")

```

```python
ax.set_xscale("log")
ax.set_yscale("log")
ax.legend()
ax.set_xlabel(r"Scale factor $a$")
ax.set_ylabel(r"Energy density $\rho$")
secax = ax.secondary_xaxis('top')
secax.set_xticks(plot_scale_factors)
secax.set_xticklabels(plot_temperatures_neutrino)
secax.set_xlabel("Neutrino Temperature [eV]")
fig.savefig("p4_all_densities.png",dpi=300,bbox_inches='tight')
fig.show()

"""
Now show ratio
"""
massless_case = (21/8)*(4/11)**(4/3)

fig, ax = plt.subplots()

ax.plot(scale_factors,sum(rho_n)/rho_g,label=r"$\rho _{\nu} / \rho _{\gamma }
    $",color="navy")
ax.axhline(y=massless_case,color="rebeccapurple",linestyle="--",label="
    Massless case")
ax.set_xscale("log")
ax.legend()
ax.set_xlabel(r"Scale factor $a$")
ax.set_ylabel(r"Ratio")

secax = ax.secondary_xaxis('top')
secax.set_xticks(plot_scale_factors)
secax.set_xticklabels(plot_temperatures_neutrino)
secax.set_xlabel("Neutrino Temperature [eV]")
fig.savefig("p4_ratios.png",dpi=300,bbox_inches='tight')
fig.show()
```