# ASTR 425/525
# Homework 4 solutions

Fall 2025

Due November 10th

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# Problem 1 (6 points)

In class, we argued that the neutron freeze-out occurs at a temperature of $T_\mathrm{f} \simeq 0.8$ MeV. This temperature can be estimated by comparing the weak interaction rate of the neutrons with the Hubble rate at that epoch. The interaction rate for the key reactions

$$\begin{aligned} p + \overline{\nu}_\mathrm{e} &\leftrightarrow n + e^+ \\ p + e^- &\leftrightarrow n + v_\mathrm{e} \end{aligned} \tag{1}$$

is given by

$$\Gamma_\mathrm{W}(x) = \left(\frac{255}{\tau_\mathrm{n}}\right)\frac{12 + 6x + x^2}{x^5} \tag{2}$$

where $\tau_\mathrm{n} = 886.7$ seconds is the neutron lifetime, and

$$x = Q/T \tag{3}$$

$$Q \equiv m_\mathrm{n} - m_\mathrm{p} = 1.2933 \text{ MeV} \tag{4}$$

On the other hand, the Hubble expansion rate can be gotten from the Friedmann equation

$$H^2 = \frac{8\pi G}{3}\rho_\mathrm{rad} \tag{5}$$

$$\rho_\mathrm{rad} = \frac{\pi^2}{30} g_*(T) T^4 \tag{6}$$

Heuristically, neutron freeze-out will occur at when $\Gamma_\mathrm{W} \sim H$. A more precise value for $T_\mathrm{f}$ can be obtained by solving

$$\Gamma_\mathrm{W}(T_\mathrm{f}) = \frac{3}{2} H(T_\mathrm{f}) \tag{7}$$

Using the expressions given above, show that $T_\mathrm{f} \simeq 0.8$ MeV.

Perhaps, the easiest way to do that is to plot both $\Gamma_\mathrm{W}(T)$ and $H(T)$ and determine where they intersect. Be mindful of the units to make sure you are comparing the two rates in the same unit system. What value of $g_*(T)$ should you use in the above?

At the time of freeze-out, the bath of relativistic species included electrons, positrons, neutrinos, and photons, so

$$g_* = 10.75 \tag{8}$$

Besides finding $x$ by inspection, we can use a numerical root solver, since the task of finding $T_f$ as $\Gamma_\mathrm{W}(T_\mathrm{f}) = \frac{3}{2} H(T_\mathrm{f})$ is equivalent to asking at what temperature does the function

$$f(x) = \Gamma_\mathrm{W}(T_\mathrm{f}) - \frac{3}{2} H(T_\mathrm{f}) \tag{9}$$

attain a value of zero. Here's one approach for units:

- Keep $\tau_\mathrm{n}$ in seconds, so that $\Gamma_\mathrm{W}$ is in $\mathrm{s}^{-1}$.
- Write $\rho$ as

$$\rho = \frac{\pi^2}{30} g_* \frac{Q^4}{x^4} \tag{10}$$

so that if $Q$ is in MeV then $\rho$ is in MeV$^4$.

- Compute Hubble using the reduced Planck mass (which we write in MeV, see Baumann eq. 3.2 for more details)

$$M_{\text{Pl}} = \sqrt{\frac{\hbar c}{8\pi G}} = 2.4 \times 10^{21} \text{ MeV} \tag{11}$$

So that

$$H(x) = \sqrt{\frac{\rho(x)}{3M_{\text{Pl}}^2}} \tag{12}$$

- The Hubble rate above is in MeV. To convert to $s^{-1}$, we use the fact that

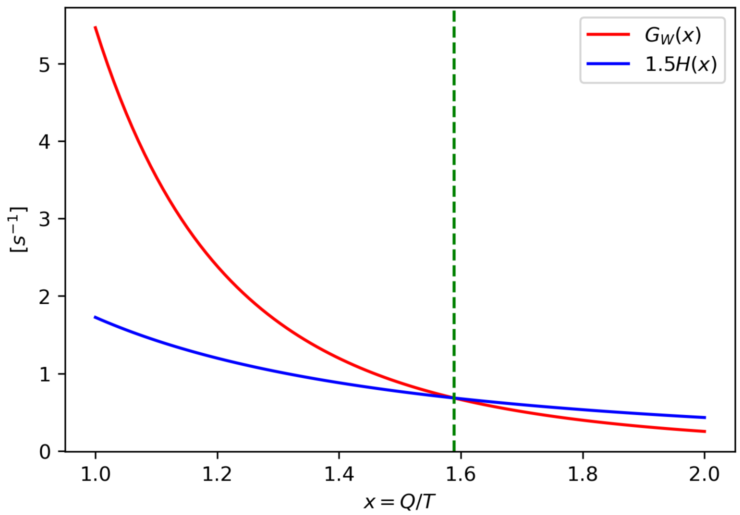
$$\hbar = 6.582 \times 10^{-22} \text{ MeV} \cdot \text{s} \tag{13}$$

The code below numerically finds the root of $f(x)$, as and produces a plot for visual confirmation.

```python
import numpy as np
from scipy.optimize import fsolve
import matplotlib.pyplot as plt

# Constants
tau_n = 886.7                 # neutron lifetime in seconds
Q = 1.2933                    # MeV
M_Pl_red = 2.4e21             # Reduced Planck mass in MeV
g_star = 10.75
hbar = 6.582e-22              # MeV*s

def Gamma(x): # (in s^-1)
    return (255/tau_n) * (12 + 6*x + x**2)/(x**5)

def H(x): # (in s^-1)
    rho = (np.pi**2 / 30) * g_star * (Q**4 / x**4)
    H_MeV = np.sqrt(rho / (3 * M_Pl_red**2))          # MeV
    return H_MeV / hbar             # s^-1

# Finding when Gamma = (3/2)H is the same as
# when Gamma - (3/2)H=0
def f(x): # where x=Q/T, so we still need to find T once we find x.
    return Gamma(x) - 1.5 * H(x)
# We will use scipy to numerically find the zero of this function.

# Scipy's fsolve needs an initial guess. Most numbers work, but we know
# we should expect something around x = Q/0.8 MeV ~ 1.6
initial_guess = 1.6
x_f = fsolve(f, initial_guess)[0]

T_f = Q / x_f
print(f"     Zero found at x = {x_f:.5f}")
print(f"Corresponding to T_f = {T_f:.5f} MeV")

x_values = np.linspace(1,2,100)
Gammas = [Gamma(xx) for xx in x_values]
Hs = [1.5*H(xx) for xx in x_values]

```

```
plt.plot(x_values, Gammas, color="red",label=r"$G_W(x)$")
plt.plot(x_values, Hs, color="blue",label=r"$1.5 H(x)$")
plt.axvline(x=x_f,color="green",linestyle="--")
plt.xlabel(r"$x=Q/T$")
plt.ylabel(r"[$s^{-1}$]")
plt.legend(loc="upper right")

plt.savefig("Q1_plot.png",dpi=300, bbox_inches='tight')
```

Results follow:




```
Zero found at x = 1.58902
Corresponding to T_f = 0.81390 MeV
```

# Problem 2 (6 points)

In class, we mentioned several times that the age of the Universe was about 1 second when neutrons froze out at $T_\mathrm{f} = 0.8$ MeV. Let us derive this result. First, remember that the age of the universe at scale factor $a$ is given (exactly) by

$$t(a) = \int_0^a \frac{da'}{a'H(a')} \tag{14}$$

The issue is that we don't know $H(a)$ but instead we know $H(T)$:

$$\begin{aligned} H(T) &= \sqrt{\frac{8\pi G}{3}\rho_\mathrm{rad}(T)} \\ \rho_\mathrm{rad}(T) &= \frac{\pi^2}{30} g_*(T) T^4 \end{aligned} \tag{15}$$

To make matters worse, $T$ does not scale as $1/a$ when $g_*$ (or $g_{*S}$) is changing. However, we can derive an approximate expression that is pretty accurate by taking $g_*(T)$ to be constant and $T \propto 1/a$.

a) Show that if $T \propto 1/a$, then

$$\frac{dT}{T} = -\frac{da}{a} \tag{16}$$

b) Using the above and assuming $g_*(T)$ to be constant, show that the age of the universe at temperature $T$ was

$$\frac{t}{\mathrm{sec}} \simeq \frac{2.42}{\sqrt{g_*}} \left(\frac{T}{\mathrm{MeV}}\right)^{-2} \tag{17}$$

c) Using the appropriate value for $g_*$ for $T \sim 1$ MeV, show that the age of the Universe at $T_\mathrm{f} = 0.8$ MeV was $t 1.15$ seconds.

a) Let $k$ be the constant of proportionality: $T = k/a$, then

$$\begin{aligned} dT &= -\frac{k}{a^2} da \\ dT &= -\frac{k}{a} \cdot \frac{da}{a} \\ dT &= -T\frac{da}{a} \\ \frac{dT}{T} &= -\frac{da}{a} \end{aligned} \tag{18}$$

b) Note that $k$ is, effectively, a *normalization* constant. As such, we can absorb it/set it to 1.

Observe that we can rewrite Eq. 14 as

$$\begin{aligned} t &= -\int_{\infty}^{1/a} \frac{dT'}{T' H(T')} \\ &= -\int_{\infty}^{1/a} \frac{dT'}{\frac{\pi\sqrt{g_*}}{M_{\mathrm{Pl,red}}\sqrt{90}} T'^3} \\ &= \frac{M_{\mathrm{Pl,red}}\sqrt{90}}{\pi\sqrt{g_*}} \frac{a^2}{2} \\ &= \frac{M_{\mathrm{Pl,red}}\sqrt{90}}{\pi\sqrt{g_*}} \frac{1}{2T^2} \ \mathrm{MeV}^{-1} \\ &= \frac{M_{\mathrm{Pl,red}}\hbar\sqrt{90}}{\pi\sqrt{g_*}} \frac{1}{2T^2} \ \text{seconds} \end{aligned} \tag{19}$$

Consider a direct computation of the prefactor:

$$\begin{aligned} \frac{\sqrt{90}}{2\pi} M_{\mathrm{Pl,red}}\hbar &= \frac{\sqrt{90}}{2\pi}\left(2.43\times 10^{21}\ \mathrm{MeV}\right)\left(6.582\times 10^{-22}\ \mathrm{MeV}\cdot\mathrm{s}\right) \\ &= 2.4199 \simeq 2.42 \end{aligned} \tag{20}$$

So (observing that this formula only works if we feed $T$ in MeV)

$$\frac{t}{\mathrm{sec}} \simeq \frac{2.42}{\sqrt{g_*}}\left(\frac{T}{\mathrm{MeV}}\right)^{-2} \tag{21}$$

c) At 1 MeV, $g_* = 10.75$, so

$$\begin{aligned} t &= \frac{2.42}{\sqrt{10.75}}(0.8)^{-2} \ \text{seconds} \\ &= 1.153 \text{ seconds} \end{aligned}$$

# Problem 3 (3 points)

Use the Saha equation for the free electron fraction $X_e$,

$$\left(\frac{1 - X_e}{X_e^2}\right)_{eq} = \frac{2\zeta(3)}{\pi^2}\eta_b\left(\frac{2\pi T}{m_e}\right)^{3/2} e^{B_H/T} \tag{22}$$

to show that the temperature of the universe when 90% of the electrons have combined with protons to form neutral atoms (that is, $X_e = 0.1$) is $T_{rec} \simeq 0.3$ eV. Why is $T_{rec} \ll B_H = 13.6$ eV?

We can find $T_{rec}$ numerically by finding the location of the root of[1]

$$f(T) = \frac{1 - 0.1}{0.1^2} - \frac{2\zeta(3)}{\pi^2}\eta_b\left(\frac{2\pi T}{m_e}\right)^{3/2} e^{B_H/T} \tag{23}$$

We find that

```
T_rec = 0.29610 eV
```

using the following code:

```python
import numpy as np
from scipy.optimize import fsolve

# Constants
zeta_three = 1.20206
eta_b = 6.0e-10
mass_e = 5.11e5       # eV
B_H = 13.6            # eV

def Saha_LHS(X_e):
    return (1-X_e)/(X_e**2)

def Saha_RHS(T):
    return 2*zeta_three * eta_b * (2*np.pi*T/mass_e)**(3/2)*np.exp(B_H/T) / (np.
   pi**2)

# Finding when T when X_e = 0.1
def f(T):
    return Saha_LHS(0.1) - Saha_RHS(T)
# We will use scipy to numerically find the zero of this function.

# Scipy's fsolve needs an initial guess. Most numbers work, but we know
# we should expect something around T ~ 0.3 eV
initial_guess = 0.3
T_rec = fsolve(f, initial_guess)[0]

print(f"T_rec = {T_rec:.5f} eV")
```

To see why $T_{red} \ll B_H = 13.6$ eV, we must recall that this temperature does not dictate the exact energy of photons, but rather their energetic distribution. At $T = 13.6$ eV, the universe still has sufficiently energetic photons that are capable of ionizing Hydrogen. By dropping the temperature, we are suppressing this tail in the photon distribution, hence the smaller value for $T_{rec}$.

[1]Where the constants are $m_e = 5.11 \times 10^5$ eV, $\eta_b = 6 \times 10^{-10}$, $B_H = 13.6$ eV.

# Problem 4 (3 points)

Even in the absence of recombination (that is, $X_e = n_e/(n_p + n_H) = 1$ at all times), the photons populating the our Universe would eventually decouple from the baryons anyway due to the volumetric dilution from the expansion.

Estimate the temperature and redshift at which photon decoupling would occur in a Universe that is always ionized. Use

$$
\begin{aligned}
H_0 &= 2.133h \times 10^{-33}\ \mathrm{eV} && (24)\\
h &= 0.674 && (25)\\
\Omega_m &= 0.315 && (26)\\
\eta_b &= 6.1 \times 10^{-10} && (27)\\
\sigma_T &= 1.71 \times 10^{-3}\ \mathrm{MeV}^{-2} && (28)
\end{aligned}
$$

The process that's keeping photons and electrons coupled is Thomson scattering

$$e^- + \gamma \to e^- + \gamma \tag{29}$$

With interaction rate

$$\Gamma = n_e \sigma_T \tag{30}$$

(where $\sigma_T$, the Thomson cross section, is given above). Decoupling occurs when the interaction rate becomes smaller than the expansion rate $H$ of the universe, so $T_{\rm dec}$ is the temperature that satisfies

$$\Gamma(T_{\rm dec}) \sim H(T_{\rm dec}) \tag{31}$$

We start by rewriting $\Gamma$ using the fact that $n_e = n_b X_e$ and the assumption that $X_e = 1$:

$$
\begin{aligned}
\Gamma &= n_e \sigma_T \\
&= n_b \cancel{X_e} \sigma_T \\
&= \eta n_\gamma \sigma_T \\
&= \eta \frac{2\zeta(3)}{\pi^2} \sigma_T T^3
\end{aligned} \tag{32}
$$

Giving us an expression for $\Gamma$ that depends only on $T$ and known constants. Similarly with the Hubble rate $H$:

$$
\begin{aligned}
H &= H_0 \sqrt{\Omega_m a^{-3}} \\
&= H_0 \sqrt{\Omega_m} \left( \frac{T}{T_0} \right)^{3/2}
\end{aligned} \tag{33}
$$

Where we assume a matter-dominated universe. We can analytically solve for $T$:

$$
\begin{aligned}
\eta \frac{2\zeta(3)}{\pi^2} \sigma_T T^3 &= H_0 \sqrt{\Omega_m} \left( \frac{T}{T_0} \right)^{3/2} \\
T_{\rm dec} &= \left( \frac{\pi^2 H_0 \sqrt{\Omega_m}}{2\zeta(3) \eta \sigma_T T_0^{3/2}} \right)^{2/3} \\
&= 9.19 \times 10^{-9}\ \mathrm{MeV}
\end{aligned} \tag{34}
$$

And redshift

$$\begin{aligned} T &= (1+z)T_0 \\ 1+z &= \frac{T}{T_0} \\ z_{\text{dec}} &= \frac{T}{T_0} - 1 \\ &= 38.1214 \end{aligned} \tag{35}$$

A much later time compared to the standard $z_{\text{dec}} \sim 1090$.