ASTR 425/525 Cosmology
Cosmic Microwave Background: Anisotropies and Power Spectra

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I. CMB ANISOTROPIES

[**Discovery of the CMB:**] Last time, we discussed the cosmic microwave background (CMB) as the most perfect blackbody ever measured, with a temperature today $T_0 = 2.725$ K. This nearly uniform radiation from the early Universe was first observed serendipitously in 1964 by Penzias and Wilson from Bell Lab (Nobel Prize in 1978). Other groups were actively looking for this radiation at the time, most notably Wilkinson, Roll, and Dicke at Princeton University, who were essentially scooped by Penzias and Wilson's discovery.

[**CMB anisotropies: Motion Dipole**] Now, we have said multiple times in this class that the CMB temperature is uniform across the sky. This is not technically correct, for two reasons. The first is rather trivial and not very important in the grand scheme of things: the Earth, the solar system, and our galaxy are all moving with respect to the CMB rest frame (the frame in which the temperature would be uniform across the sky). This leads to large dipole in the CMB temperature: the photons coming from our direction of motion are slightly hotter (more energy) than those coming from the direction we are moving away from. This dipole can be written as

$$\frac{\delta T(\hat{\mathbf{n}})}{\bar{T}} \equiv \frac{T(\hat{\mathbf{n}}) - \bar{T}}{\bar{T}} = \hat{\mathbf{n}} \cdot \mathbf{v}, \tag{1}$$

where $T(\hat{\mathbf{n}})$ is the temperature of the CMB in the $\hat{\mathbf{n}}$ direction, $\bar{T}$ is the average CMB temperature across the sky ($= T_0$ today), and $\mathbf{v}$ is our velocity with the respect to the CMB rest frame. This velocity has $|\mathbf{v}| = 368$ km/s in the direction of the crater constellation. This dipole is thus about a 0.12% correction to the otherwise nearly uniform CMB temperature.

[**CMB anisotropies: Primordial fluctuations**] The second reason is a lot more interesting as it connects to the physics of the early Universe: the Universe was not *exactly* homogeneous when photons last scattered around redshift $z_* \simeq 1100$. Indeed, both the radiation and matter densities were not completely uniform in the early Universe: there were places where the photon-baryon plasma was a little hotter than average and places where the plasma was a little colder than average (similar for the (dark) matter density: some places were denser, some places were less dense). The origin of these primordial fluctuations is a very important topic in and of itself; we will come back to those during our discussion of cosmological inflation.

[**CMB anisotropies: Main sources**] Here, we will assume that these fluctuation are there and sketch their impact on what we observe today. Let us define the photon density contrast

$$\delta_\gamma(\boldsymbol{x}, \eta) \equiv \frac{\rho_\gamma(\boldsymbol{x}, \eta) - \bar{\rho}_\gamma(\eta)}{\bar{\rho}_\gamma(\eta)}, \tag{2}$$

where $\bar{\rho}_\gamma(\eta)$ is the (spatially) average photon density of the Universe at conformal time η. Here, when $\delta_\gamma(\boldsymbol{x}, \eta) > 0$, it means that the photons at point $\boldsymbol{x}$ and time η are slightly hotter than average. The opposite is true for $\delta_\gamma < 0$. The presence of non-vanishing δ_γ at last scattering source a non-zero $\delta T/\bar{T}$. Since $\rho_\gamma \propto T^4$, we have that

$$\frac{\delta T(\hat{\mathbf{n}})}{\bar{T}} \simeq \left.\frac{\delta_\gamma}{4}\right|_*, \tag{3}$$

where the $*$ mean that we need to evaluate δ_γ on the surface of photon last scattering at redshift $z_* \simeq 1100$. Now, after they last scatter, photons have to climb out (of fall off) any gravitational potentials that might be present at last scattering. If a photon has to climb out of a gravitational potential $\Psi < 0$ (corresponding to an overdensity), the photon will lose energy in the process, contributing to $\delta T/\bar{T} < 0$. If the potential is positive $\Psi > 0$, corresponding to an underdense region), the photon will gain energy while escaping the potential after last scattering ($\delta T/\bar{T} > 0$). It turns out that the temperature change of the photons is exactly equal to the depth of the gravitational potential they were in at last scattering. Adding this contribution, we now have

$$\frac{\delta T(\hat{\mathbf{n}})}{\bar{T}} \simeq \left.\left(\frac{\delta_\gamma}{4} + \Psi\right)\right|_*, \tag{4}$$

where again we need to evaluate the RHS at last scattering. These two terms together describe the so-called Sachs-Wolfe (SW) effect. There is another source of temperature anisotropies cause by the motion of the electrons on which photons last scatter: if these electrons were moving towards us at large scattering, this boosts the temperature of the photon because of the apparent Doppler shift. An opposite effect occurs for electrons moving away from us. The temperature contrast then becomes

$$\frac{\delta T(\hat{\mathbf{n}})}{\bar{T}} \simeq \left(\frac{\delta_\gamma}{4} + \Psi\right)\bigg|_* - \hat{\mathbf{n}} \cdot \mathbf{v}_b|_* \tag{5}$$

where the leading minus sign ensures that the contribution to $\delta T/\bar{T}$ is positive when $\mathbf{v}_b$ is pointing at us (i.e. $\hat{\mathbf{n}} \cdot \mathbf{v}_b < 0$). Here, we have used $\mathbf{v}_b$ (the baryon velocity) instead of $\mathbf{v}_e$ since the two are equal (because of strong Coulomb interactions). This last term is referred to as the Doppler term. Finally, photons can encounter changing gravitational potentials on their way between last scattering and us (static gravitational potentials have no impact). Dynamics gravitational potentials make a difference since a photon can gain energy by falling into one, but then doesn't have to spend that energy getting out of it if the potential has decayed by then. This term depends on the integral of Ψ', the time derivative of the gravitational potential. Adding its contribution leads to

$$\frac{\delta T(\hat{\mathbf{n}})}{\bar{T}} \simeq \left(\frac{\delta_\gamma}{4} + \Psi\right)\bigg|_* - \hat{\mathbf{n}} \cdot \mathbf{v}_b|_* + \int_{\eta_*}^{\eta_0} d\eta(2\Psi'), \tag{6}$$

where η_* is the conformal time at photon last scattering and η_0 is the conformal time today. The last term denotes the *integrated Sachs-Wolfe effect*. Taken together, these three terms characterize the primary CMB anisotropies (a detailed calculation would include a few more contributions).

II. EVOLUTION OF PHOTON FLUCTUATIONS

To compute the CMB anisotropies, we thus need to know what δ_γ, Ψ, $\mathbf{v}_b$, etc. are doing near photon last scattering. In general, computing this would require numerically solving a bunch of coupled differential equations. Here, let's just look at the evolution of δ_γ to get a sense of how that would work. The evolution of photon density fluctuations are governed by two elements: pressure and gravity. This can be summarized by this equation

$$\delta_\gamma' = -\frac{4}{3}\theta_\gamma + 4\Psi', \tag{7}$$

where the first term on the right-hand side is the pressure fluctuation $\theta_\gamma = -\boldsymbol{k} \cdot \mathbf{v}_\gamma$, a prime denotes derivative with respect to conformal time, and Ψ is the Newtonian gravitational potential. The pressure term itself obeys this equation

$$\theta_\gamma' = k^2\left(\frac{1}{4}\delta_\gamma - \sigma_\gamma\right) + k^2\Psi + a n_{\rm e}\sigma_T(\theta_{\rm b} - \theta_\gamma), \tag{8}$$

where $\theta_{\rm b} = -\boldsymbol{k} \cdot \mathbf{v}_{\rm b}$ is the velocity of the baryon fluid. σ_γ is the photon anisotropic stress. For photon to have significant anisotropic stress, they need to be able to travel significant distances between scattering. However, in the early Universe, the mean free path of photon between collisions with electron is tiny and $\sigma_\gamma \sim 0$ at early times. The baryons obey a similar set of equation

$$\delta_b' = -\theta_b + 3\Psi' \tag{9}$$

$$\theta_b' = -\frac{a'}{a}\theta_b + k^2\Psi + \frac{a n_{\rm e}\sigma_T}{R}(\theta_\gamma - \theta_{\rm b}) \tag{10}$$

where $R = 3\rho_b/(4\rho_\gamma)$. Thomson scattering between photons and free electrons and very efficient before recombination (i.e. $n_e\sigma_T/H \gg 1$), which sets $\theta_\gamma \approx \theta_b$. We can combine the two photon equations as

$$\delta_\gamma'' + \frac{k^2}{3}\delta_\gamma \approx 0, \tag{11}$$

where we have neglected the source term involving the gravitational potential. This is just the equation of an harmonic oscillator. It's solution are sound wave propagating in the photon-baryon plasma with sound speed $c_s = 1/\sqrt{3}$

$$\delta_\gamma(k,\eta) = A\cos\left(k\eta/\sqrt{3}\right) + B\sin\left(k\eta/\sqrt{3}\right), \tag{12}$$

where A and B are constant set by the initial conditions. It turns out that $B = 0$ in our Universe and only the first term survives. This solution neglects the presence of the baryons; including them at early times would change the sound speed to

$$c_s = \frac{1}{\sqrt{3(1+R)}}. \tag{13}$$

Since $R > 1$ near last scattering, including the baryons reduces the sound speed of the plasma. This is called the baryon-loading effect.

In the real Universe, we have a lot of these sound waves propagating in all different directions. At last scattering, the combination all these waves get imprinted on the last scattering surface.

III. CMB POWER SPECTRA

The cosmic microwave background (CMB) temperature anisotropies on the celestial sphere can be decomposed into spherical harmonics

$$\Theta(\hat{\boldsymbol{n}}) = \frac{\delta T(\hat{\boldsymbol{n}})}{\bar{T}} = \sum_{l=2}^{\infty}\sum_{m=-1}^{l} a_{lm} Y_{lm}(\hat{\boldsymbol{n}}), \tag{14}$$

where $\hat{\boldsymbol{n}} = \hat{\boldsymbol{n}}(\theta,\phi)$ is a unit vector on the sphere, $\bar{T} = \langle T(\hat{\boldsymbol{n}})\rangle$ is the average temperature of the CMB sky, and Y_{lm} are spherical harmonics, which form a complete basis for functions defined on the unit sphere. Here, $\langle \ldots \rangle$ denotes the ensemble average over different realizations of the CMB sky. We also have removed the dipole ($l = 1$) since it is dominated by our motion through space rather than the primary anisotropies. In general, we cannot predict the exact structure of the CMB sky, but rather its two-point correlation function $C(\theta)$

$$C(\theta) = \langle \Theta(\hat{\boldsymbol{n}})\Theta(\hat{\boldsymbol{n}}')\rangle, \tag{15}$$

where $\cos\theta = \hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{n}}'$. This is usually phrased in terms of the a_{lm} coefficients

$$\langle a_{lm} a^*_{l'm'}\rangle = C_l \delta_{ll'}\delta_{mm'}, \tag{16}$$

where the Kronecker deltas are a consequence of statistical isotropy. Here, C_l is referred to as the *angular temperature power spectrum*. This is the most common way we use to describe the statistical properties of the CMB.