ASTR 425/525 Cosmology
Distance Units

(Dated: August 20, 2025)

I. THE PARSEC

The Universe is so vast that the distance unit systems we use on Earth (or even in the Solar system) are awfully inadequate to capture the humongous typical distances involved in cosmology. To remedy this, we introduce the *parsec*. The definition of the parsec is anchored on the average Sun-Earth distance (which is about 1.496×10^{11}m) and is defined as follows. An object (like a star) is at a distance of exactly 1 parsec if a right triangle made of the Sun, Earth, and this object has an opening angle of 1 arcsecond (see Fig. 1). In case you don't know, 1 arcsecond is 1/3600 of a degree or 4.848×10^{-6} radian. Working our the basic geometry of this triangle, we have that

$$1\,\text{parsec} = 3.086 \times 10^{16}\,\text{m}. \tag{1}$$

In term of lightyears (the distance travel by light in a year), we have 1 parsec = 3.261 lightyears. Even though a parsec (pc) is a large distance, in cosmology we will typically use *kiloparsecs* (=10^3 pc) and *megaparsecs* (=10^6 pc), abbreviated "kpc" and "Mpc", respectively.

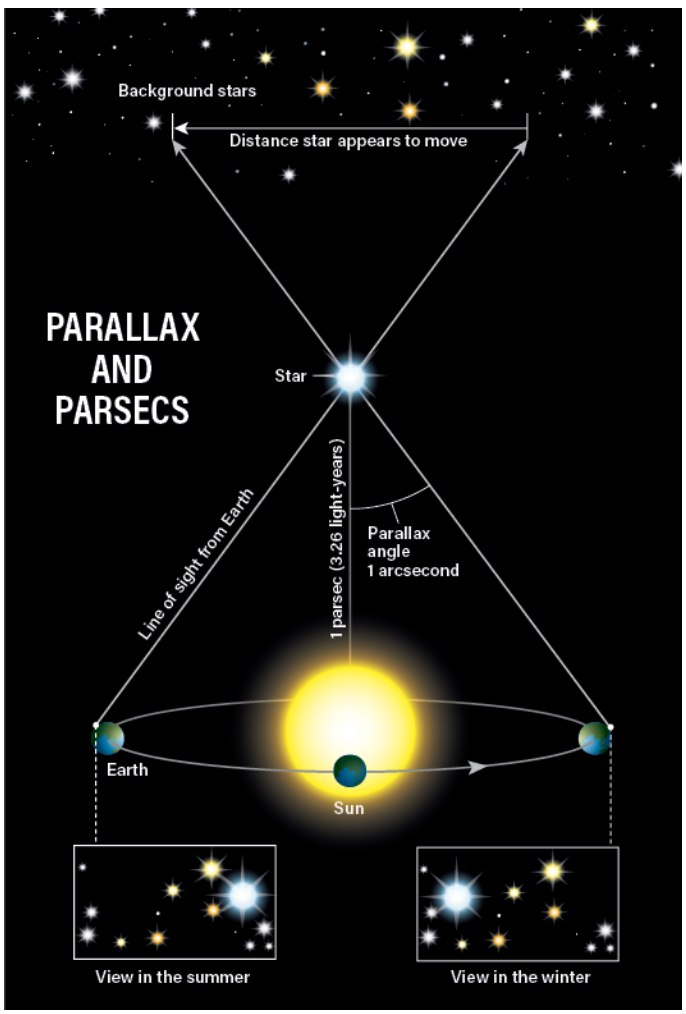

FIG. 1. The definition of a parsec. Image credits: Astronomy Magazine.

Here are some useful numbers to keep in mind:

- Solar system distance from the center of our galaxy: ~ 8 kpc.
- Size of the Milky Way galactic disk: ~ 30 kpc.
- Distance to the Large Magellanic Cloud: ~ 48.5 kpc.
- Distance to the Andromeda galaxy (our nearest neighbor): ~ 770 kpc.
- Typical distance between large galaxies in our Universe: ~ 1 Mpc.
- Distance over which the Universe becomes homogeneous: ~ 100 Mpc
- Comoving size of the observable Universe: ~ 14 Gpc $= 14,000$ Mpc .

II. DISTANCES AND THE METRIC

The Universe is vast and expanding (more on that later). Being able to rigorously define distances in such a Universe is extremely important. As we do in much of physics when we want to describe a quantitative phenomenon, we first set up a coordinate system. In three-dimensional space, a familiar coordinate system is the cartesian one, where every point in space is labelled by a triplet of numbers (x, y, z). The coordinate distance between two points $\mathbf{r} = (x, y, z)$ and $\mathbf{r}' = (x', y', z')$ in this space is the familiar

$$\Delta s^2 = |\mathbf{r}' - \mathbf{r}|^2 = (x' - x)^2 + (y' - y)^2 + (z' - z)^2. \tag{2}$$

Often, we are interested in the coordinate distance between two points that are separated by an infinitesimal amount. For instance, taking $\mathbf{r}' = (x + dx, y + dy, z + dz)$, the distance between $\mathbf{r}$ and $\mathbf{r}'$ is

$$ds^2 = dx^2 + dy^2 + dz^2. \tag{3}$$

Note how this infinitesimal distance element does not depend on the coordinates themselves, that is, it is independent of where you are in this three dimensional space. This implies that three-dimensional Euclidean space is homogeneous and isotropic. I want to make an important distinction between the left-hand side and the right-hand side of Eq. (3). The LHS is a physical (squared) length scale that any observer would agree on. On the other hand, the RHS is a coordinates-dependent (squared) distance. We are free to rewrite this RHS in another coordinate system. For instance, we can use spherical coordinates (r, θ, ϕ) to write the same infinitesimal line element as above as

$$ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{4}$$

Note that in this case, the infinitesimal line element is not just the square the coordinate differences between the two nearby points (that is, $ds^2 \neq dr^2 + d\theta^2 + d\phi^2$); Instead, the distance depends explicitly on r and θ. This should make intuitive sense: if you travel a small angle $d\phi$ in the $\hat{\boldsymbol{\phi}}$, the actual physical distance traversed depends on your radial coordinates r and the polar angle θ. Note that this is still standard Euclidean geometry, just written in a funny coordinate system. But the spherical coordinates example illustrates an important point: *coordinate* distances are generally not equivalent to *physical* distances. To map coordinate distances to physical distances, we need a very special object: the *metric*.

The metric is a mathematical object that defines a notion of distance on vector spaces. Sticking to three-dimensional space here, we can generally write the infinitesimal line element as

$$ds^2 = \sum_{i,j=1}^{3} g_{ij} du^i du^j, \tag{5}$$

where u^i are the coordinates used to chart the space, and g_{ij} is the metric. Basically, you can think of the metric written here as a 3×3 matrix. For example, in cartesian coordinates we have $u^1 = x$, $u^2 = y$, and $u^3 = z$, and metric components are $g_{11} = g_{22} = g_{33} = 1$, with all other elements being zero. In other words, the metric in this case is nothing more than the identity matrix $g_{ij} = \delta_{ij}$, where δ_{ij} is the Kronecker delta. In spherical coordinates, we have $u^1 = r$, $u^2 = \theta$, and $u^3 = \phi$, and the metric elements are $g_{11} = 1$, $g_{22} = r^2$, and $g_{33} = r^2 \sin^2\theta$, with all other elements zero.

III. DISTANCES IN COSMOLOGY

In cosmology, we do not necessarily care about the distance between two points in three-dimensional space, but we care *a lot* about the distance between two points (or events) in four-dimensional *spacetime*. Using cartesian coordinates, any point in such spacetime can be labelled by the tuple (ct, x, y, z), where c is the speed of light introduced to make sure that the four entries all have the same units. In this course, we will always work in natural units where $c = 1$, so we will always write this tuple as (t, x, y, z).

Generalizing the Eq. (5) above to a *spacetime* interval ds^2 as

$$ds^2 = \sum_{\mu,\nu=0}^{3} g_{\mu\nu} dx^\mu dx^\nu, \tag{6}$$

with the index 0 referring to the "time" component, and $1 - 3$ index refers to the "spatial" components. In flat non-expanding spacetime (which is the spacetime that you've been using so far in most physics classes), the spacetime interval, written using spatial cartesian coordinates, is

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2. \tag{7}$$

The only funny business here is that the time component has an overall negative sign. We will typically write this as

$$ds^2 = \sum_{\mu,\nu=0}^{3} \eta_{\mu\nu} dx^\mu dx^\nu, \tag{8}$$

where

$$\eta_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{9}$$

is referred to as the *Minkowski* metric. In this course, we will use $\eta_{\mu\nu}$ exclusively to denote the Minkowski metric, while $g_{\mu\nu}$ will be used to denote the more general metric of (usually curved) spacetime. Note that because of the intimate relationship between ds^2 and $g_{\mu\nu}$ given in Eq. (6), we sometime refer to ds^2 as the metric itself, which is a slight abuse of language.

Now in cosmology, we do not live in a Minkowski spacetime. For one, we know the Universe is expanding. We also need our spacetime metric to reflect the cosmological principle, which states that the Universe is, on average, homogeneous and isotropic. As we mentioned in the previous section, three-dimensional Euclidean space is homogeneous and isotropic, we only need a small modification to the Minkowski metric to describe a smooth, expanding Universe. Using spatial cartesian coordinates, the most general metric I can write down is

$$ds^2 = -f(t, x, y, z)dt^2 + g(t, x, y, z)dx^2 + h(t, x, y, z)dy^2 + l(t, x, y, z)dz^2. \tag{10}$$