ASTR 425/525 Cosmology
The Fluid Equation

(Dated: September 3, 2025)

I. CONSERVATION OF ENERGY

Last time, we introduced the *Hubble rate*

$$H \equiv \frac{\dot{a}}{a}, \tag{1}$$

where $a(t)$ is the scale factor characterizing the expansion (or contraction) of the Universe, and an overhead dot denotes a derivative with respect to coordinate time t. The question we would like to answer now is what determines the time evolution of H. We would do a detailed derivation of this evolution next week, but for now we will just refer to a central idea in *General Relativity* by quoting physicist Sir John Archibald Wheeler:

"Spacetime tells matter how to move; matter tells spacetime how to curve."

What this tells us is that the matter/energy content of the Universe will determine the behavior of the Hubble rate H. Indeed, H is related to the curvature of *spacetime* in a universe described by a FLRW metric. Schematically, we will have that $H^2(t) \propto \rho_{\rm tot}(t)$, where $\rho_{\rm tot}(t)$ is the total energy density of the Universe, which include contributions from all the matter, radiation, dark energy, etc. in the Universe.

To determine $H(t)$, we will thus need to know how $\rho_{\rm tot}(t)$ changes with time, that is, we need an evolution equation for all different energy components $\rho_I(t)$ making up the Universe. To derive such an equation, we start from the first law of thermodynamics, which is simply a statement about energy conservation.

$$dE + pdV = TdS, \tag{2}$$

where E is the total energy, p is the pressure, V is the physical volume, T is the temperature, and S is the entropy. Assuming an adiabatic reversible evolution, which is actually an excellent approximation for most important processes going on in cosmology, we set $dS = 0$ here. Now consider a fluid with energy density $\rho(t)$. This means $E = \rho(t)V(t)$. Now, we can write the physical volume V as

$$V(t) = a^3(t)V_{\rm com}, \tag{3}$$

where $V_{\rm com}$ is a fixed comoving volume. In a small time interval dt, the change in energy dE is then,

$$dE = (\dot{\rho}a^3 + 3\rho a^2\dot{a})V_{\rm com}dt \tag{4}$$

where we have used

$$dV = d(a^3V_{\rm com}) = 3a^2\dot{a}V_{\rm com}dt. \tag{5}$$

The first law of thermodynamics then takes the form

$$(\dot{\rho}a^3 + 3\rho a^2\dot{a})V_{\rm com}dt + 3pa^2\dot{a}V_{\rm com}dt = 0. \tag{6}$$

Dividing everything by $a^3V_{\rm com}dt$, we get

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0, \tag{7}$$

which is the desired evolution equation for $\rho(t)$. Note that this equation involves yet another quantity, the pressure p. To close the system of equation, we thus need an *equation of state* w relating the pressure to the energy density of whichever fluid we are considering,

$$w \equiv \frac{p}{\rho}. \tag{8}$$

While the equation of state w could technically have some time dependence, we will focus for now on components with constant equation of state. Some of these are listed in Table I. In this case, one can derive a simple expression for how the energy density ρ depends on the scale factor a of the Universe.

TABLE I. Equation of state for different possible components of the Universe.

Component type	w
kinetic energy	1
radiation	1/3
non-relativistic matter	0
spatial curvature	-1/3
Cosmological constant	-1

II. AN ILLUSTRATIVE EXAMPLE

Recent measurements of the baryonic acoustic oscillation (BAO) signal from the Dark Energy Spectroscopic Instrument (DESI) suggest that dark energy might not be a cosmological constant ($w = -1$), but instead could have a time-dependent equation of state given by

$$w(a) = w_0 + w_a(1 - a), \tag{9}$$

where w_0 and w_a are constants. Even in this case, we can analytically solve the continuity equation for the evolution of this strange dark energy component.

$$\begin{aligned}
\frac{d\rho}{dt} &= -3\rho \frac{(1 + w_0 + w_a(1 - a))}{a} \frac{da}{dt} \\
\frac{d\rho}{\rho} &= -3\left(1 + w_0 + w_a(1 - a)\right) \frac{da}{a} \\
\int \frac{d\rho}{\rho} &= -3\left(1 + w_0 + w_a\right) \int \frac{da}{a} + 3w_a \int da \\
\ln \rho &= -3\left(1 + w_0 + w_a\right) \ln a + 3w_a a + C \\
\ln \rho &= \ln a^{-3(1+w_0+w_a)} + 3w_a a + C,
\end{aligned} \tag{10}$$

where C is a constant of integration. Exponentiating on both sides, we obtain

$$\rho = K a^{-3(1+w_0+w_a)} e^{3w_a a}, \tag{11}$$

where $K = e^C$. Demanding that $\rho(a = 1) = \rho_0$ (the dark energy density today), we finally obtain

$$\rho = \rho_0 a^{-3(1+w_0+w_a)} e^{3w_a(a-1)}. \tag{12}$$

Fit to the data indicates that $w_0 \simeq -0.7$ and $w_a \simeq -1$, which results in a dark energy component with an energy density that is *increasing* (see Fig. 1 below) as the Universe expands until $a \sim 0.7$, before slowly decreasing until the present time ($a = 1$). This is a very strange behavior as we would naively expect energy densities to dilute as the Universe expands (or, as a limiting case, to stay constant), not increase. It is very difficult to build realistic physical models for such a behavior.

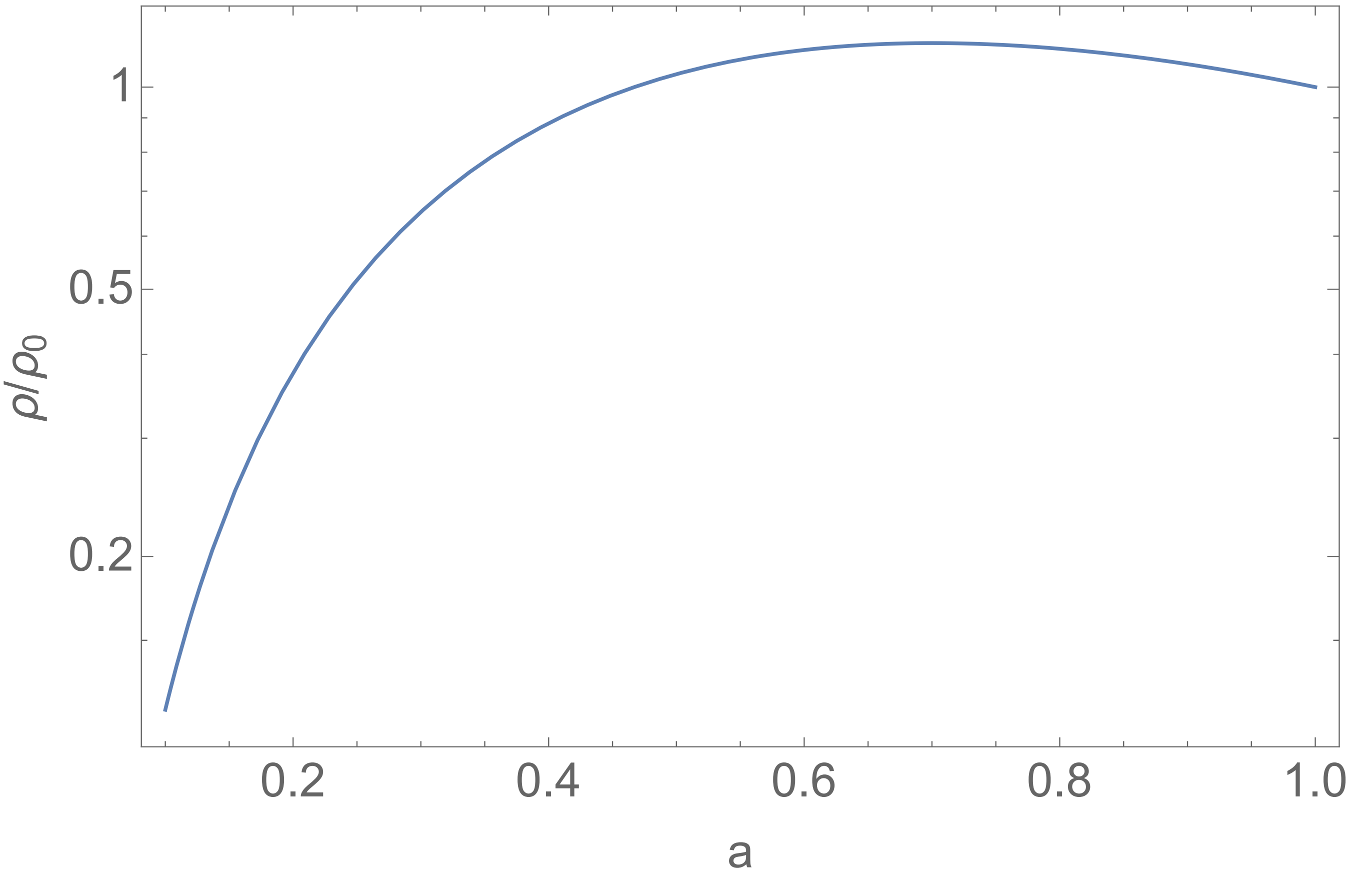

FIG. 1. The evolution of the energy density for dynamical dark energy.