ASTR 425/525 Cosmology
The Friedmann Equation and Curvature

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I. THE FRIEDMANN EQUATION

We have argued that the cosmological principle leads to the Friedmann-Lemaître-Robertson-Walker metric, which in its spatially flat version (and written in cartesian coordinates) takes the form

$$ds^2 = -dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right), \tag{1}$$

where $a(t)$ is the scale factor. We would now like to compute the evolution of the scale factor $a(t)$. This technically requires General Relativity, which fundamentally relates the energy content of the Universe to its geometry and evolution. However, it turns out that a calculation based on Newtonian mechanics leads to essentially the same result, so we will follow that route instead.

A key element of Newtonian gravity is that the gravitational field (or acceleration) $\mathbf{g}$ obeys Gauss's law: If I integrate $\mathbf{g}$ over a closed surface S, the gravitational "flux" going through the surface depends only on the mass enclosed within this surface $M_{\rm enc}$, or mathematically

$$\oint_S \mathbf{g}\cdot d\mathbf{a} = -4\pi G M_{\rm enc}. \tag{2}$$

This is, of course, very similar to Gauss's law for the electric field, and is ultimately the consequence of both fields obeying a $1/r^2$ law. Note that the result above doesn't depend on the details of the surface S, as long at encloses the same mass $M_{\rm enc}$. For example, the gravitational field (or acceleration) a distance r away from a point mass M located at the origin is

$$\mathbf{g} = -\frac{GM}{r^2}\hat{\mathbf{r}}, \tag{3}$$

and using a sphere of radius R as the surface S ($d\mathbf{a} = R^2 d\Omega\hat{\mathbf{r}}$, with $d\Omega = \sin\theta d\theta d\phi$), we obtain

$$\begin{aligned}\oint_S \mathbf{g}\cdot d\mathbf{a} &= -\int \frac{GM}{R^2}\hat{\mathbf{r}}\cdot R^2 d\Omega\hat{\mathbf{r}}\\ &= -GM\int d\Omega\\ &= -4\pi GM,\end{aligned} \tag{4}$$

which verifies the above as M is the only mass enclosed in this case. Since any finite mass distribution can be built by adding together point masses, the above can be generalized to arbitrary mass distributions. In particular, consider a universe filled with a uniform matter density $\rho(t)$. Pick an arbitrary origin in this universe (since it is assumed to be homogeneous, all choices of origin must be equivalent, and our final answer will not depend on this choice), and consider a test mass m a distance r away from that origin. Since the gravitational field obeys Gauss's law, only the mass contained within the sphere of radius r surrounding the origin can exert a force on the test mass m. This mass is simply

$$M(t) = \int \rho(t) d^3r = \frac{4\pi r^3\rho(t)}{3}, \tag{5}$$

since the matter density $\rho(t)$ is uniform in space.

To get the evolution of the scale factor $a(t)$, we will examine the evolution of the energy of the test particle m. This particle has both gravitational potential energy and kinetic energy. Remembering that the gravitational potential Φ is related to the gravitational field by $\mathbf{g} = -\boldsymbol{\nabla}\Phi$, the gravitational potential at the location of the mass m is simply

$$\Phi(t) = -\frac{GM(t)}{r} = -\frac{4\pi G r^2\rho(t)}{3}. \tag{6}$$

Since the gravitational potential is the potential energy per unit mass, the potential energy V of the particle m is then simply

$$V = -\frac{4\pi G r^2 \rho(t) m}{3}. \tag{7}$$

Meanwhile, the kinetic energy of the particle m is

$$T = \frac{1}{2} m \left(\frac{dr}{dt}\right)^2. \tag{8}$$

Then, the total energy U of this particle is then

$$U = T + V = \frac{1}{2} m \left(\frac{dr}{dt}\right)^2 - \frac{4\pi G r^2 \rho(t) m}{3}. \tag{9}$$

Here, r is a physical distance between the origin and the mass m. We now introduce comoving coordinates $\mathbf{r}_{\rm com}$

$$\mathbf{r} = a(t)\mathbf{r}_{\rm com}, \tag{10}$$

such that

$$\frac{dr}{dt} \equiv \dot{r} = \dot{a} r_{\rm com}. \tag{11}$$

The total energy is then

$$U = \frac{1}{2} m \dot{a}^2 r_{\rm com}^2 - \frac{4\pi G a^2 r_{\rm com}^2 \rho(t) m}{3}. \tag{12}$$

Dividing both sides by $a^2 r_{\rm com}^2 m/2$, we get

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2 = \frac{8\pi G}{3}\rho(t) - \frac{k}{a^2}, \tag{13}$$

where we have defined $k = -\frac{2U}{r_{\rm com}^2 m}$. Here, k is a constant with units of [length]$^{-2}$ whose physical meaning is a little mysterious at this point. As we will soon see, this constant is related to the spatial geometry of the Universe. It turns out that setting $k = 0$ correspond to having the flat FLRW metric given in Eq. (1) above.

Equation (13) is called the *Friedmann equation*. It is one of the most important equations in all of cosmology. It relates the energy density of the Universe to the behavior of the scale factor, such that universes dominated by different kind of energy (matter, radiation, dark energy) will behave differently. To solve the Friedmann equation, we generally need to know the how $\rho(t)$ changes with time, that is, we need an evolution equation for $\rho(t)$ to close this system of equations. Last time, we derived the continuity equation governing the evolution of $\rho(t)$ so we can now solve for the scale factor.

II. CURVATURE

As we mentioned above, the constant k has units of [length]$^{-2}$. What is the length scale appearing here? To clarify this, it is useful to write $k \equiv \kappa/R^2$, where $\kappa = \{-1, 0, 1\}$, and R is a constant with units of [length]. To obey the cosmological principle, this length scale has to be the same at every point in spacetime. There is only one possibility: R stands for the global spatial curvature. That is, at any instant in time, the Universe as a whole can be a manifold (3D surface) with a constant curvature. There are only 3 possibilities for such a manifold (corresponding of course to $\kappa = \{-1, 0, 1\}$): a flat Euclidean manifold ($\kappa = 0$, or equivalently $R \to \infty$), a spherical manifold ($\kappa = 1$), and an hyperbolic manifold ($\kappa = -1$).

So far, we only have dealt with the spatially flat FLRW metric (which corresponds to $\kappa = 0$). How does the metric change when $\kappa \neq 0$? It actually takes the very simple form

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1 - \kappa(r/R)^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]. \tag{14}$$

For $r/R \ll 1$, this reduces to the standard flat case, showing that the curvature only starts mattering once $r \sim R$. It is informative to introduce a new radial coordinate χ via

$$d\chi \equiv \frac{dr}{\sqrt{1-\kappa(r/R)^2}}. \tag{15}$$

Using integration, this definition can be inverted for r. Defining $u = r/R$, we can write

$$\chi = R\int_0^{r/R} \frac{du}{\sqrt{1-\kappa u^2}} = \begin{cases} R\sinh^{-1}(r/R) & \text{if } \kappa = -1, \\ r & \text{if } \kappa = 0, \\ R\sin^{-1}(r/R) & \text{if } \kappa = +1. \end{cases} \tag{16}$$

Inverting these relations, we obtain

$$r = \begin{cases} R\sinh(\chi/R) & \text{if } \kappa = -1, \\ \chi & \text{if } \kappa = 0, \\ R\sin(\chi/R) & \text{if } \kappa = +1. \end{cases} \tag{17}$$

Sometime, we use the above relations to write the FLRW metric as

$$ds^2 = -dt^2 + a^2(t)\left[d\chi^2 + S_\kappa^2(\chi)(d\theta^2 + \sin^2\theta d\phi^2)\right]. \tag{18}$$

with

$$S_\kappa(\chi) = \begin{cases} R\sinh(\chi/R) & \text{if } \kappa = -1, \\ \chi & \text{if } \kappa = 0, \\ R\sin(\chi/R) & \text{if } \kappa = +1. \end{cases} \tag{19}$$

III. GEOMETRIES

Let's now consider the three geometries in turn:

- **Flat (Euclidean) geometry** ($\kappa = 0$): In this case, the metric is that given in Eq. (1) above. This geometry is characterized by:

  1. The inner angles of a triangle add up to π.
  2. The circumference of a circle of radius r is $2\pi r$.
  3. Two lines that are initially parallel will stay parallel forever.

- **Closed (spherical) geometry** ($\kappa = +1$): In this case, the metric takes the form

  $$ds^2 = -dt^2 + a^2(t)\left[d\chi^2 + R^2\sin^2(\chi/R)d\Omega^2\right], \tag{20}$$

  where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. Defining $q \equiv \chi/R$, this can be rewritten as

  $$ds^2 = -dt^2 + a^2(t)R^2\left[dq^2 + \sin^2 q d\Omega^2\right], \tag{21}$$

  whose spatial part is the metric of a three-sphere of radius $a(t)R$. Since it is difficult to visualize a three-sphere, let's focus on its equatorial plane by setting $\theta = \pi/2$. At a fixed coordinate time t, we then have

  $$ds^2 = (aR)^2(dq^2 + \sin^2 q d\phi^2), \tag{22}$$

  which the familiar metric of a two-sphere of radius aR. We can then think of q as the "polar" angle of this sphere. Drawing a circle on this sphere at a constant polar angle q, the radius of this circle (as measured along the surface of the sphere) is $r = aRq$. However, the circumference of this circle is $2\pi aR\sin q < 2\pi r$. This is not Euclidean geometry, and this space is said to be (positively) *curved*, with radius of curvature R.

  Looking at Eq. (20), we see that when $\chi \ll R$, the metric appears appears approximately flat since $R^2\sin^2(\chi/R) \to \chi^2$ in this case. Only when χ becomes a sizable fraction of the radius of curvature R that we start "feeling" the curvature. This makes sense for us living on Earth: we don't really see that the Earth is roughly round in our everyday lives since we only see a very small area of it at any given time. But if you go to space and see the whole planet at once, it is easy to see it is roughly a sphere.

  In cosmology, we refer to this case with positive curvature as an *closed universe*, since at any given time, the Universe is a giant three-sphere and thus have a finite volume. This geometry is is characterized by:

1. The inner angles of a triangle add up to more than π.
2. The circumference of a circle of radius r is less than $2\pi r$.
3. Two lines that are initially parallel will eventually converge.

- **Open (hyperbolic) geometry** ($\kappa = -1$): In this case, the metric takes the form

$$ds^2 = -dt^2 + a^2(t)\left[d\chi^2 + R^2 \sinh^2(\chi/R) d\Omega^2\right], \tag{23}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. The spatial part of this metric represents a three-dimensional hyperboloid (saddle) with constant negative curvature $-R$. Much of our discussion for the spherical case applies here, except with the substitution $\sin \to \sinh$. In particular, the circumference of a circle of radius $r = aRq$ (as measured along the saddle; $q = \chi/R$ as before) is $2\pi aR \sinh q > 2\pi r$.

In cosmology, we refer to this case with negative curvature as an *open universe*, since at any given time, the Universe is infinite. This geometry is is characterized by:

1. The inner angles of a triangle add up to less than π.
2. The circumference of a circle of radius r is more than $2\pi r$.
3. Two lines that are initially parallel always diverge from each other.