ASTR 425/525 Cosmology
Hubble rate and redshift

(Dated: August 27, 2025)

I. THE HUBBLE RATE

Last time, we saw the FLRW metric, which describes the homogeneous and isotropic expansion of the Universe

$$ds^2 = -dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right), \tag{1}$$

or in spherical coordinates

$$ds^2 = -dt^2 + a^2(t)\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]. \tag{2}$$

where $a(t)$ is the scale factor describing the expansion (or contraction) of the Universe. Remember that we have set the speed of light $c = 1$ here. Here, the spatial coordinates (x, y, z) or r are *comoving* coordinates. These coordinates form a global spatial grid, usually defined at the current epoch (t_0). The scale factor uniformly stretches this grid as the Universe expand. Physical distances are then given by

$$r_{\rm phys}(t) = a(t) r_{\rm com}, \tag{3}$$

where $r_{\rm com}$ is the comoving distance separating two objects. Defining the comoving grid today means that $a(t_0) = 1$. The physical velocity of an object in the expanding Universe can then be written as

$$\mathbf{v}_{\rm phys} = \frac{d\mathbf{r}_{\rm phys}}{dt} = \frac{da}{dt}\mathbf{r}_{\rm com} + a(t)\frac{d\mathbf{r}_{\rm com}}{dt} = \frac{1}{a}\frac{da}{dt}\mathbf{r}_{\rm phys} + \mathbf{v}_{\rm pec} = H\mathbf{r}_{\rm phys} + \mathbf{v}_{\rm pec}. \tag{4}$$

Here,

$$H \equiv \frac{\dot{a}}{a} \tag{5}$$

is the *Hubble rate* (or sometime called Hubble parameter) describing the expansion rate of the Universe. Here, an overhead dot denotes d/dt. The first term $H\mathbf{r}_{\rm phys}$ denotes the *Hubble flow*, describing how objects (like galaxie) in the Universe recedes from each other due to the expansion of the cosmos. The second term $\mathbf{v}_{\rm pec} = a(t)\dot{\mathbf{r}}_{\rm com}$ denotes the *peculiar velocity*, which is the velocity measured by an observer following the Hubble flow. It corresponds to the motion of objects (e.g. galaxies) with respect to the comoving grid, usually caused by their mutual gravitational attraction.

The current value of the Hubble rate $H(t_0)$ is a very important object in cosmology: the usually referred to it as the *Hubble constant* and denote it H_0. It's current value in the Universe today is about $H_0 \simeq 70$ km/s/Mpc. For objects separated by large enough distances, peculiar velocities are small and their observed receding speeds observed from Earth today are approximately

$$v_{\rm phys} \approx H_0 r_{\rm phys}. \tag{6}$$

This is called the Hubble-Lemaître law, which states that the receding speed of objects in the Universe is proportional to their distance from us. With the value of H_0 given above, objects separated by 1 Mpc are receding from each other at a speed of 70 km/s. Closer objects recede less fast, while more distant objects recede faster.

Today, objects situated at distances greater than $r_{\rm phys} = 1/H_0$ (or c/H_0 if you restore the factor of speed of light) appear to be moving away from us faster than the speed of light. This does not contradict special relativity since this is only an apparent relative velocity, and no inertial observer is seeing objects moving faster than the speed of light. However, this distance is so important that we give is a special name: the *Hubble radius* $r_H \equiv 1/H_0$. This distance tells us about the typical size of the Universe today.

II. REDSHIFT

In the previous section, we discussed how the receding speed of objects in the Universe is proportional to their physical distance from us. But how do we measure receding speeds of objects in cosmology? We do so by measuring the wavelength of light coming from distant objects. Light propagating in an expanding Universe has its wavelength stretched by the expansion. We characterize this via the redshift z, which is defined as

$$z \equiv \frac{\lambda_{\rm obs} - \lambda_{\rm em}}{\lambda_{\rm em}} \tag{7}$$

where $\lambda_{\rm obs}$ is the observed wavelength and $\lambda_{\rm em}$ is wavelength at emission. The wavelength is a physical distance (between two wave crests, say) and thus scales as any physical distance, that is, $\lambda \propto a$. We thus have

$$\frac{\lambda_{\rm obs}}{\lambda_{\rm em}} = \frac{a(t_{\rm obs})}{a(t_{\rm em})} \tag{8}$$

You may ask how do we know the wavelength at emission $\lambda_{\rm em}$? Astronomers look at the spectrum of light coming from distant objects and find pattern of *atomic lines* (like the Lyman-α line of hydrogen) which have known wavelengths here on Earth. Typically, the observation time is simply $t_{\rm obs} = t_0$, and this reduces to $\lambda_{\rm obs}/\lambda_{\rm em} = 1/a(t_{\rm em})$, since $a(t_0) = 1$. This means that the redshift is simply related to the scale factor by

$$z = \frac{1}{a} - 1 \tag{9}$$

or in a more compact form

$$a = \frac{1}{1+z}. \tag{10}$$

Since $a(t_0) = 1$ today, this means that $z = 0$ today, and higher redshifts denote earlier times (with $z = \infty$ denoting the Big Bang). Just like a, z is often used as a time-variable, remembering that redshifts run backward. For instance, astronomers frequently use redshift to denote how old an object they are looking at is.