ASTR 425/525 Cosmology
Thermodynamics for Cosmologists II

(Dated: October 1, 2025)

I. NUMBER DENSITY, ENERGY DENSITY, AND PRESSURE

Last time, we introduced the particle distribution function (or phase-space density) $f(\boldsymbol{x}, \boldsymbol{p}, t)$ describing the occupancy of different position and momentum eigenstates. We then used the Cosmological Principle to argue that the phase space density must be independent of position and cannot depend on the direction of the momentum $\boldsymbol{p}$, that is,

$$f(\boldsymbol{x}, \boldsymbol{p}, t) = f(p, t), \tag{1}$$

where $p = |\boldsymbol{p}|$. In the cosmological context, the time dependence of the phase-space density typically enters via the *temperature* T since the Universe tends to cool as it expands. In fact, it is rather natural to use the temperature as the time variable (much like we have used the scale factor a or the redshift z as a time variable). Here, we will adopt this approach and write

$$f(p, t) = f(p, T(t)) = f(p, T). \tag{2}$$

Using this notation, we can review the key quantities we introduced last time, including the number density

$$n(T) = g \int \frac{d^3p}{(2\pi)^3} f(p, T), \tag{3}$$

which is the number of particle of a given species per unit volume. Here, g is the number of internal degrees of freedom. Remember that we have set $\hbar = 1$ here. We also introduced the energy density

$$\rho(T) = g \int \frac{d^3p}{(2\pi)^3} f(p, T) E(p). \tag{4}$$

where $E(p) = \sqrt{p^2 + m^2}$ is the energy. In the above, we have assumed that the particles are essentially free, that is, that we can neglect the interaction energies between the particles. This is usually a very good approximation in cosmology. Meanwhile, the pressure P was given by

$$P(T) = g \int \frac{d^3p}{(2\pi)^3} f(p, T) \frac{p^2}{3E(p)}. \tag{5}$$

We also discussed that if a particle species is in kinetic equilibrium (i.e. particles are able to efficiently exchange energy and momentum), then the particle distribution function takes either a *Fermi-Dirac* or *Bose-Einstein* form

$$f(p, T) = \frac{1}{e^{(E(p)-\mu)/T} \pm 1}, \tag{6}$$

where the $+$ sign is for fermions (half-integer spin) and the $-$ sign for bosons (integer spin). Here, μ is the *chemical potential.* Here, we have set the Boltzmann constant $k_\mathrm{B} = 1$, meaning that we are measuring temperature in units of energy (eV, say). In cosmology, if a species has a nonzero chemical potential, it means that the number of particles and of the corresponding anti-particles are different. For a particle species X and its anti-particle $\bar{X}$, we generally have

$$\mu_X = -\mu_{\bar{X}}. \tag{7}$$

II. RELATIVISTIC LIMIT

A. Number density

Let us first consider the relativistic limit $p \gg m$, such that $E \simeq p$. Here, we set the chemical potential to zero. The number density is then given by

$$\begin{aligned} n(T) &= g \int \frac{d^3p}{(2\pi)^3} \frac{1}{e^{p/T} \pm 1} \\ &= \frac{g}{2\pi^2} \int_0^\infty dp \frac{p^2}{e^{p/T} \pm 1} \\ &= \frac{gT^3}{2\pi^2} \int_0^\infty dx \frac{x^2}{e^x \pm 1}, \end{aligned} \tag{8}$$

where we have changed the variable to $x = p/T$. We can now use the known result that

$$\int_0^\infty dx \frac{x^n}{e^x - 1} = \zeta(n+1)\Gamma(n+1), \tag{9}$$

where $\zeta(z)$ is the Riemann zeta function and $\Gamma(n+1) = n!$ (for integer n) is the gamma function. For bosons, we immediately get the result

$$n_{\text{Bosons}}(T) = \frac{g\zeta(3)T^3}{\pi^2}, \tag{10}$$

where $\zeta(3) \approx 1.202$. For fermions, we can also use the above result once we notice that

$$\frac{1}{e^x + 1} = \frac{1}{e^x - 1} - \frac{2}{e^{2x} - 1}, \tag{11}$$

which allows us to write

$$\begin{aligned} \int_0^\infty dx \frac{x^2}{e^x + 1} &= \int_0^\infty dx \left[\frac{x^2}{e^x - 1} - \frac{2x^2}{e^{2x} - 1} \right] \\ &= 2\zeta(3) - 2\int_0^\infty \frac{dy}{2} \frac{(y/2)^2}{e^y - 1} \\ &= 2\zeta(3)\left(1 - \frac{1}{4}\right) \\ &= 2\zeta(3)\frac{3}{4}. \end{aligned} \tag{12}$$

For fermions, we thus get

$$n_{\text{Fermions}}(T) = \frac{3}{4}\frac{g\zeta(3)T^3}{\pi^2}. \tag{13}$$

We thus obtain the general behavior that $n \propto T^3$ for a species in thermal equilibrium. For example, the number density of CMB photons today ($g = 2$ for the two polarization, $T_0 = 2.725\text{K} = 2.348\times10^{-4}$ eV, [using $k_B = 8.617\times10^{-5}$ eV/K]) is given by

$$n_\gamma(t_0) = \frac{2\zeta(3)T_0^3}{\pi^2} \simeq 410\,\text{photons/cm}^3, \tag{14}$$

where we have used $\hbar c = 1.97 \times 10^{-5}$ eV cm.

B. Energy density

Using $E(p) = p$, the energy density takes the form

$$\begin{aligned}\rho(T) &= g\int \frac{d^3p}{(2\pi)^3}\frac{p}{e^{p/T}\pm 1}\\ &= \frac{g}{2\pi^2}\int_0^\infty dp \frac{p^3}{e^{p/T}\pm 1}\\ &= \frac{gT^4}{2\pi^2}\int_0^\infty dx \frac{x^3}{e^x \pm 1},\end{aligned} \tag{15}$$

where we have defined $x = p/T$. For bosons, we can directly used the result from Eq. (9) to obtain

$$\rho_{\rm Bosons}(T) = 3!\frac{gT^4}{2\pi^2}\zeta(4) = 3\frac{gT^4}{\pi^2}\frac{\pi^4}{90} = g\frac{\pi^2}{30}T^4, \tag{16}$$

where we have used the fact that $\zeta(4) = \pi^4/90$. For fermions, we use the same trick as in Eq. (11), which allows us to write

$$\begin{aligned}\int_0^\infty dx \frac{x^3}{e^x+1} &= \int_0^\infty dx \left[\frac{x^3}{e^x-1} - \frac{2x^3}{e^{2x}-1}\right]\\ &= 3!\zeta(4) - 2\int_0^\infty \frac{dy}{2}\frac{(y/2)^3}{e^y-1}\\ &= 6\frac{\pi^4}{90} - \frac{1}{8}6\frac{\pi^4}{90}\\ &= 2\frac{7}{8}\frac{\pi^4}{30}.\end{aligned} \tag{17}$$

Thus, for fermions the energy density is

$$\rho_{\rm Fermions}(T) = g\frac{7}{8}\frac{\pi^2}{30}T^4. \tag{18}$$

We this obtain the general behavior that $\rho \propto T^4$ for a species in thermal equilibrium. At the same temperature, a fermionic species has an energy density that is suppressed by a factor of 7/8 compared to a similar bosonic gas.

For example, the energy density in CMB photons today is

$$\rho_\gamma(t_0) = 2\frac{\pi^2}{30}T_0^4 = \frac{\pi^2}{15}T_0^4 = 2.0\times 10^{-15}\,{\rm eV}^4. \tag{19}$$

Dividing this by the critical density of the Universe today $\rho_{\rm c} = 3H_0^2/(8\pi G) = 8.098h^2 \times 10^{-11}\,{\rm eV}^4$, we get

$$\Omega_\gamma = \frac{\rho_\gamma(t_0)}{\rho_{\rm c}} = 2.47\times 10^{-5}h^{-2}, \tag{20}$$

which is the number we quoted before. Here, h is the reduced Hubble rate $h = H_0/(100\,{\rm km/s/Mpc})$.

C. Pressure

Using $E(p) = p$, the pressure takes the form

$$P(T) = g\int \frac{d^3p}{(2\pi)^3}f(p)\frac{p}{3} = \frac{1}{3}g\int \frac{d^3p}{(2\pi)^3}f(p)p = \frac{\rho(T)}{3}, \tag{21}$$

that is, we just retrieve the standard equation of state for relativistic particles $w = P/\rho = 1/3$.