Question 1

a) As we have just seen, the energy density for radiation (relativistic particles) in thermal equilibrium scales as

$$\rho_{\rm rad} \propto T^4 \tag{1}$$

Use your knowledge of how radiation energy density dilutes in an expanding universe to determine how the temperature scales with the scale factor $a(t)$ of the universe.

b) Given your answer from part (a), how does the number density $n_{\rm rad}$ of relativistic particles scale with $a(t)$? Does this answer make sense? Why?

c) Does radiation need to be in thermal equilibrium to have an equation of state $w = P/\rho = 1/3$? Why or why not?

a) We start by recalling that for radiation ($w = 1/3$), we have

$$\rho(a) \propto a^{-4} \tag{2}$$

(see the solution to Worksheet 5 for details). We piece this with the relation between energy density and temperature, and observe see that:

$$\begin{aligned} T &\propto (\rho_{\rm rad})^{1/4} \\ &\propto (a^{-4})^{1/4} \\ &= a^{-1} \\ &= \frac{1}{a} \end{aligned} \tag{3}$$

So

$$T \propto \frac{1}{a} \tag{4}$$

b) Recall that

$$n(t) \propto T^3$$

(for both relativistic Fermions and Bosons). So

$$n(t) \propto a^{-3} \tag{5}$$

Which makes sense. As the universe expands, the same number of particles occupy a larger volume, giving a lower number density.

c) No, this result is independent of whether radiation is in thermal equilibrium. Recall that

$$\begin{aligned} \rho(t) &= g\int \frac{d^3p}{(2\pi)^3} f(p,t) E(p) \\ P(t) &= g\int \frac{d^3p}{(2\pi)^3} f(p,t) \frac{p^2}{3E(p)} \end{aligned} \tag{6}$$

In the relativistic limit, $E \simeq p$, so

$$\begin{aligned}
P &= g \int \frac{d^3p}{(2\pi)^3} f(p,t) \frac{p^2}{3E(p)} \\
\text{apply relativistic limit...} \qquad &= g \int \frac{d^3p}{(2\pi)^3} f(p,t) \frac{E^2}{3E} \\
&= \frac{1}{3} g \int \frac{d^3p}{(2\pi)^3} f(p,t) E \\
&= \frac{1}{3} \rho
\end{aligned} \tag{7}$$

So

$$\frac{P}{\rho} = \frac{1}{3} \tag{8}$$

Regardless of the form of $f(p,t)$.