Question 1

a) We saw that the number density of non-relativistic particles with zero chemical potential in thermal equilibrium is

$$n_{\rm NR} = g\left(\frac{mT}{2\pi}\right)^{3/2} e^{-m/T} \tag{1}$$

For $T \ll m$, this tells us that the abundance of such particle is exponentially suppressed. What is the actual physical process that is responsible for exponentially suppressing the abundance?

b) Assuming that the process you found in part (a) is efficient, abundances will indeed get exponentially suppressed. However, the presence of a chemical potential can change that. Show that the difference in abundances between a particle (number density n) and its antiparticle (number density $\overline{n}$) is

$$n - \overline{n} = 2g\left(\frac{mT}{2\pi}\right)^{3/2} e^{-m/T}\sinh\frac{\mu}{T} \tag{2}$$

Where μ can be a function of temperature.

c) Near the epoch of recombination ($T \sim 1$ eV) just before neutral atoms formed, the Universe was full of non-relativistic electrons ($m_e \sim 0.5$ MeV), apparently contradicting Eq. (1) above. How is this possible? Which eventual value of the chemical potential do you need to make this possible?

a) In the case of $\mu = 0$, the number densities for the particle and its antiparticle are equal to each other.

As such, and for practical purposes, the particle-antiparticle annihilation process can take place freely (that is, the universe doesn't have to worry about keeping a certain asymmetry between their number densities as a result of a non-zero chemical potential).

The difference between this scenario and that of higher energies (higher T) is that a low T, producing the pair of particles becomes too "expensive" for the universe to carry out, so there is a single process taking place and this process annihilates them. This gives rise to this exponential suppression.

In other words, while pair-production "shuts down," annihilation still takes place.

b) Suppose that the chemical potential is not zero. Further, if μ_X is the chemical potential of the particle (X), then the chemical potential of the antiparticle $\overline{X}$ is

$$\mu_{\overline{X}} = -\mu_X \tag{3}$$

Recall that

$$n_{\mathrm{NR}}(T) \equiv n(T) = g\left(\frac{mT}{2\pi}\right)^{3/2} \exp\left(-\frac{m - \mu}{T}\right) \tag{4}$$

So the number density for the antiparticle is

$$\overline{n}(T) = g\left(\frac{mT}{2\pi}\right)^{3/2} \exp\left(-\frac{m + \mu}{T}\right) \tag{5}$$

Together and with some algebraic manipulations, we observe that

$$\begin{aligned}
n - \overline{n} &= g\left(\frac{mT}{2\pi}\right)^{3/2} \left(\exp\left(-\frac{m - \mu}{T}\right) - \exp\left(-\frac{m + \mu}{T}\right)\right) \\
&= g\left(\frac{mT}{2\pi}\right)^{3/2} \exp\left(-\frac{m}{T}\right) \left[\exp\left(+\mu/T\right) - \exp\left(-\mu/T\right)\right] \\
&= g\left(\frac{mT}{2\pi}\right)^{3/2} \exp\left(-\frac{m}{T}\right) 2\frac{\left[\exp\left(+\mu/T\right) - \exp\left(-\mu/T\right)\right]}{2} \\
&= g\left(\frac{mT}{2\pi}\right)^{3/2} \exp\left(-\frac{m}{T}\right) 2\sinh(\mu/T) \\
&= 2g\left(\frac{mT}{2\pi}\right)^{3/2} \exp\left(-\frac{m}{T}\right) \sinh(\mu/T)
\end{aligned} \tag{6}$$

Where the fact that $\sin(x) = \frac{e^x - e^{-x}}{2}$ was used.

c) Equation 1 assumes zero chemical potential. This is not the case in our universe! Suppose that $n \gg \overline{n}$ so that $n - \overline{n} \simeq n$. Then

$$n \simeq n - \overline{n} = 2g\left(\frac{mT}{2\pi}\right)^{3/2} \exp\left(-\frac{m}{T}\right) \sinh(\mu/T) \tag{7}$$

Observe that $m \gg T$. In this case, we must have

$$\mu \simeq m = 0.5 \text{ MeV} \tag{8}$$

To make the equations balance out correctly.