Question 1

In this question, we want to determine the equilibrium abundance of deuterium (D) nuclei at temperatures $T \sim$ a few MeV, which are assembled via the reaction

$$n + p \leftrightarrow D + \gamma \tag{1}$$

a) Assuming $\mu_\gamma = 0$ and that the above reaction is in chemical equilibrium, show that

$$\left(\frac{n_{\rm D}}{n_{\rm n} n_{\rm p}}\right)_{\rm eq.} = \frac{3}{4}\left(\frac{m_{\rm D}}{m_{\rm n} m_{\rm p}}\frac{2\pi}{T}\right)^{3/2} e^{B_{\rm D}/T} \tag{2}$$

Where $B_{\rm D} = m_{\rm n} + m_{\rm p} - m_{\rm D} = 2.2$ MeV is the binding energy of deuterium. Here, $n_{\rm n}$, $n_{\rm p}$, and $n_{\rm D}$ are the number densities of neutrons, protons, and deuterium, respectively. Note that deuterium is a spin-1 nucleus.

b) Since $B_{\rm D} \ll m_{\rm p} + m_{\rm n} \simeq 1.9$ GeV, argue that the above can be approximated written as

$$\left(\frac{n_{\rm D}}{n_{\rm p}}\right)_{\rm eq} = \frac{3}{4} n_{\rm n,eq}\left(\frac{4\pi}{m_{\rm p} T}\right)^{3/2} e^{B_{\rm D}/T} \tag{3}$$

c) Using the approximation that $n_{\rm n} \sim n_{\rm b}/2$, where $n_{\rm b}$ is the baryon number density, show that

$$\left(\frac{n_{\rm D}}{n_{\rm p}}\right)_{\rm eq} \approx 4\eta_{\rm b}\left(\frac{T}{m_{\rm p}}\right)^{3/2} e^{B_{\rm D}/T}, \tag{4}$$

where $\eta_{\rm b} \sim 6 \times 10^{-10}$ is the baryon to photon ratio. Thus, unless the exponential factor is large (which occurs for $T \ll B_{\rm D}$), the abundance of deuterium is very small.

a) We start by recalling that the number density of a given species in equilibrium is given by:

$$n_i^{\rm eq} = g_i \left(\frac{m_i T}{2\pi}\right)^{3/2} \exp\left(\frac{\mu_i - m_i}{T}\right) \tag{5}$$

Where g_i, μ_i, m_i denote the number of degrees of freedom, chemical potential, and mass of the species, respectively.

Protons and neutrons are spin 1/2 particles, so $g_{\rm p} = g_{\rm n} = 2$. Deuterium is not elementary, and happens to be spin-1, so $g_{\rm D} = 3$.[1]

Using the fact that $\mu_\gamma = 0$, we can use the reaction relation (Eq. 1) to see that

$$\mu_{\rm n} + \mu_{\rm p} = \mu_{\rm D} \tag{6}$$

[1] We combine the spin of a proton and a neutron (the constituents of Deuterium) $\frac{1}{2} \otimes \frac{1}{2}$, and we are able to get a composite system of spin 1 ("triplet") or spin 0 ("singlet"), $1 \oplus 0$. The triplet state happens to be favorable over the the singlet state.

Putting these pieces together, we can explicitly construct $\left(\frac{n_{\rm D}}{n_{\rm n}n_{\rm p}}\right)_{\rm eq.}$:

$$\begin{aligned}
\left(\frac{n_{\rm D}}{n_{\rm n}n_{\rm p}}\right)_{\rm eq.} &= \frac{3}{2\cdot 2}\left(\frac{m_{\rm D}}{m_{\rm n}m_{\rm p}}\right)^{3/2}\left(\frac{2\pi}{T}\right)^{3/2}\exp\left(\frac{\mu_{\rm D}-m_{\rm D}}{T}+\frac{m_{\rm n}-\mu_{\rm n}}{T}+\frac{m_{\rm p}-\mu_{\rm p}}{T}\right) \\
&= \frac{3}{4}\left(\frac{m_{\rm D}}{m_{\rm n}m_{\rm p}}\cdot\frac{2\pi}{T}\right)^{3/2}\exp\left(\frac{\cancel{\mu_{\rm D}-(\mu_{\rm n}+\mu_{\rm p})}\ +\ m_{\rm P}+m_{\rm n}-m_{\rm D}}{T}\right) \\
&= \frac{3}{4}\left(\frac{m_{\rm D}}{m_{\rm n}m_{\rm p}}\cdot\frac{2\pi}{T}\right)^{3/2}\exp\left(\frac{m_{\rm P}+m_{\rm n}-m_{\rm D}}{T}\right) \\
&= \frac{3}{4}\left(\frac{m_{\rm D}}{m_{\rm n}m_{\rm p}}\frac{2\pi}{T}\right)^{3/2}e^{B_{\rm D}/T}
\end{aligned} \tag{7}$$

b) Because $B_{\rm D}\ll m_{\rm p}+m_{\rm n}$, it follows that

$$m_{\rm D}\simeq m_{\rm p}+m_{\rm n}\simeq 2m_{\rm n} \tag{8}$$

Let's use this result directly in the answer from part (a):

$$\begin{aligned}
\left(\frac{n_{\rm D}}{n_{\rm n}n_{\rm p}}\right)_{\rm eq.} &= \frac{3}{4}\left(\frac{m_{\rm D}}{m_{\rm n}m_{\rm p}}\frac{2\pi}{T}\right)^{3/2}e^{B_{\rm D}/T} \\
&= \frac{3}{4}\left(\frac{2m_{\rm n}}{m_{\rm n}m_{\rm p}}\frac{2\pi}{T}\right)^{3/2}e^{B_{\rm D}/T} \\
&= \frac{3}{4}\left(\frac{1}{m_{\rm p}}\frac{4\pi}{T}\right)^{3/2}e^{B_{\rm D}/T}
\end{aligned} \tag{9}$$

So

$$\left(\frac{n_{\rm D}}{n_{\rm p}}\right)_{\rm eq.} = (n_{\rm n})_{\rm eq.}\frac{3}{4}\left(\frac{1}{m_{\rm p}}\frac{4\pi}{T}\right)^{3/2}e^{B_{\rm D}/T} \tag{10}$$

c) As given, suppose that $n_{\rm n}\sim n_{\rm b}/2$. Further, recall that

$$\begin{aligned}
n_{\rm b} &= \eta n_\gamma \quad \text{(by definition)} \\
&= \eta\frac{2\zeta(3)}{\pi^2}T^3
\end{aligned} \tag{11}$$

Substituting these expressions in the answer from part (b), we observe that

$$\begin{aligned}
\left(\frac{n_{\rm D}}{n_{\rm p}}\right)_{\rm eq.} &= (n_{\rm n})_{\rm eq.}\frac{3}{4}\left(\frac{1}{m_{\rm p}}\frac{4\pi}{T}\right)^{3/2}e^{B_{\rm D}/T} \\
&= \frac{n_{\rm b}}{2}\cdot\frac{3}{4}\left(\frac{1}{m_{\rm p}}\frac{4\pi}{T}\right)^{3/2}e^{B_{\rm D}/T} \\
&= \frac{1}{2}\cdot\left(\eta\frac{2\zeta(3)}{\pi^2}T^3\right)\frac{3}{4}\left(\frac{1}{m_{\rm p}}\frac{4\pi}{T}\right)^{3/2}e^{B_{\rm D}/T} \\
&= \frac{1}{2}\cdot\frac{2\zeta(3)}{\pi^2}\cdot\frac{3}{4}\cdot(4\pi)^{3/2}\,\eta\left(\frac{T}{m_{\rm p}}\right)^{3/2}e^{B_{\rm D}/T}
\end{aligned} \tag{12}$$

The constants yield a factor of

$$\begin{aligned} \frac{1}{2} \cdot \frac{2\zeta(3)}{\pi^2} \cdot \frac{3}{4} \cdot (4\pi)^{3/2} &= \frac{6\zeta(3)}{\sqrt{\pi}} \\ &= 4.069 \\ &\simeq 4 \end{aligned} \tag{13}$$

So

$$\begin{aligned} \left(\frac{n_{\mathrm{D}}}{n_{\mathrm{p}}}\right)_{\mathrm{eq.}} &= 4.069\, \eta \left(\frac{T}{m_{\mathrm{p}}}\right)^{3/2} e^{B_{\mathrm{D}}/T} \\ &\simeq 4\eta \left(\frac{T}{m_{\mathrm{p}}}\right)^{3/2} e^{B_{\mathrm{D}}/T} \end{aligned} \tag{14}$$