Question 1

The cosmic microwave background (CMB) is the most perfect balckbody ever observed in nature. This blackbody was measured in the early 1990s by the FIRAS instrument onboard the COBE satellite (resulting in the 2006 Nobel Prize in Physics). Starting from our knowledge of photons as bosons in thermal equilibrium, let's derive the *spectral radiation intensity* I_f (energy per unit time, per unit area, per unit frequency, per unit solid angle) that COBE measured.

a) In natural units, we know that the energy density of photons can be written as

$$\rho_\gamma(T) = g_\gamma \int \frac{d^3p}{(2\pi)^3} \frac{E(p)}{e^{E(p)/T} - 1} \tag{1}$$

where

$$E(p) = p$$
$$g_\gamma = 2$$

As we want an answer in physical units, first restore the factors of $\hbar$, c, and k_B in the above expression. Then, use the factthat the energy of a photon is $E = hf$, where h is Planck's constant ($\hbar = h/2\pi$) and f is the frequency of the photon, to write the energy density per unit frequency per unit solid angle to be

$$\frac{d\rho_\gamma}{df\, d\Omega} = 2\frac{h}{c^3} \frac{f^3}{e^{hf/k_B f} - 1} \tag{2}$$

b) Use the fact that photons always travel at the speed of light to find that

$$I_f = 2\frac{h}{c^2} \frac{f^3}{e^{hf/k_B T} - 1}$$

And sketch this function.

c) Since the CMB temperature is 2.725 today, determine the frequency f_{peak} at which the CMB blackbody peaks.

Hint: Write $x = hf/k_B T$, and solve $dI_f/dx = 0$. It's easiest to solve this iteratively.

a) Observe that the integration measure decomposes into radial + angular parts:

$$d^3p \to p^2 dp d\Omega \tag{3}$$

Further, the following quantities must be restored into physical units:

$$E \to pc \tag{4}$$
$$T \to T k_B \tag{5}$$
$$\text{Phase space state density} \quad \frac{g}{(2\pi)^3} \to \frac{g}{(2\pi\hbar)^3} = \frac{g}{h^3} \tag{6}$$

Further, $E = pc = hf$ implies that

$$p = \frac{hf}{c} \quad \Rightarrow \quad dp = \frac{h}{c} df \tag{7}$$

Noting that in the expression E will be substituted by hf, not pc, we start from

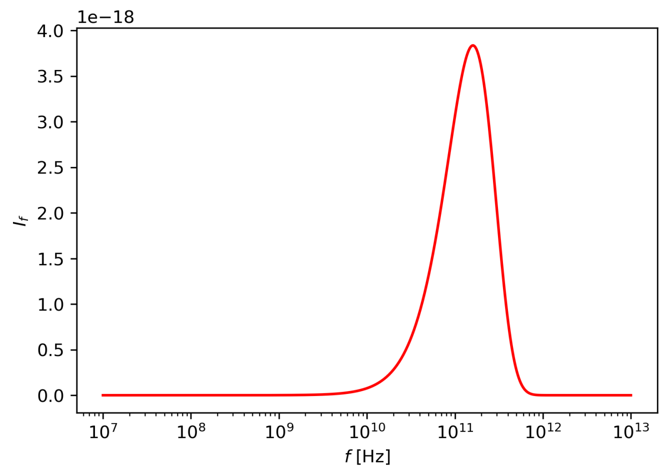
$$d\rho_\gamma = g_\gamma \frac{1}{(2\pi)^3} \frac{E}{e^{hf/K_B T} - 1} p^2 \, dp \, d\Omega \tag{8}$$

and write

$$\begin{aligned} d\rho_\gamma &= 2 \frac{1}{h^3} \frac{hf}{e^{hf/K_B T} - 1} \left(\frac{hf}{c} \right)^2 \frac{h}{c} df \, d\Omega \\ \frac{d\rho_\gamma}{df \, d\Omega} &= 2 \frac{h}{c^3} \frac{f^3}{e^{hf/k_B f} - 1} \end{aligned} \tag{9}$$

b) To match the units of I_f (as given above), we see that $\frac{d\rho_\gamma}{df d\Omega}$ is missing the units $[\text{distance}]^1 \, [\text{time}]^{-1}$, which can be introduced using the speed of photons, c. So

$$\begin{aligned} \frac{d\rho_\gamma}{df \, d\Omega} c &= c 2 \frac{h}{c^3} \frac{f^3}{e^{hf/k_B f} - 1} \\ I_f &= 2 \frac{h}{c^2} \frac{f^3}{e^{hf/k_B f} - 1} \end{aligned} \tag{10}$$



c) Let $x = hf/k_B T$. Then

$$\begin{aligned} I_f &= 2 \frac{h}{c^2} \frac{1}{\exp(x) - 1} x^3 \left(\frac{k_B T}{h} \right)^3 \\ &= y \frac{x^3}{\exp(x) - 1} \end{aligned}$$

Where $y = \frac{2k_B^3 T^3}{c^2 h^2}$. We find the minimum by setting $\frac{dI_f}{dx} = 0$ and solving for x. Observe that

$$\frac{dI_f}{dx} = -y(3 + e^x(x - 3))x^2 \frac{1}{(e^x - 1)^2} \tag{11}$$

So we need to solve:

$$3 = e^x(3 - x) \tag{12}$$

There is a more illuminating way to write this equation for x:

$$x = 3(1 - e^{-x}) \tag{13}$$

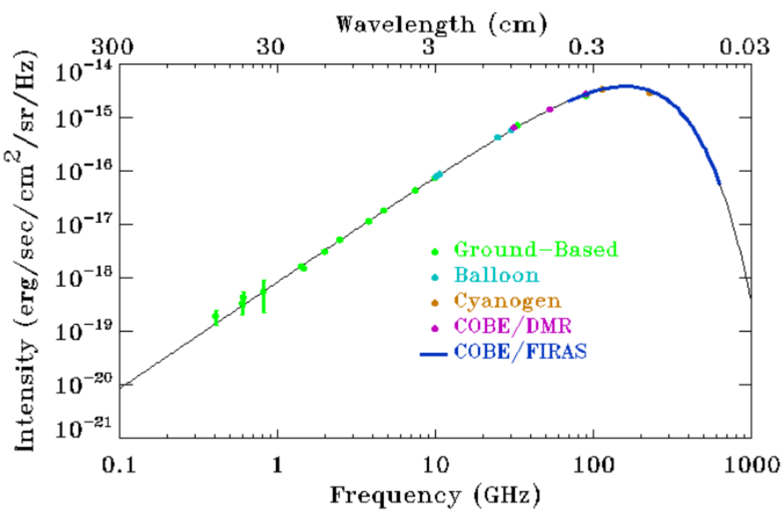
So that we can use iterations. A great initial guess is $x = 3$, since e^{-3} is quite small. Define $F(x) = 3(1 - e^{-x})$, then:

$$\begin{aligned} F(3) &= 2.85064 \\ F(2.85064) &= 2.82658 \\ F(2.82658) &= 2.82235 \\ F(2.82235) &= \boxed{2.8216} \end{aligned} \tag{14}$$

Which amounts to:

$$\begin{aligned} f &= \frac{x k_B T}{h} \\ &= 1.602 \times 10^{11} \text{ Hz} \\ &\equiv 160.2 \text{ MHz} \end{aligned} \tag{15}$$

This not only agrees with our plot in part (b), but also the observed data:



"FIRAS determined the CMB temperature to be 2.725 ± 0.001 Kelvin, with deviations from a perfect blackbody limited to less than 50 parts per million in intensity." [NASA ASD]