Problem 1

Consider a universe filled with a single component with a constant equation of state $p/\rho = w$. In this case, as we saw last time, the density of such component scales as $\rho = \rho_0 a^{-3(1+w)}$, where ρ_0 is the density today.

a) Solve the Friedmann equation (assume $k = 0$) for the scale factor $a(t)$ in such a universe, assuming that $w \neq -1$.

b) Now repeat the calculation for the case that $w = -1$.

c) How does the scale factor behave in a matter-dominated universe ($w = 0$)? How about a radiation-dominated universe ($w = 1/3$)?

a) Recall the Friedmann equation (setting $k = 0$, from the lecture notes `The Friedmann equation and curvature`, eq. 13)

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2 = \frac{8\pi G}{3}\rho(t) \tag{1}$$

Where a dot $\dot{}$ denotes a derivative with respect to time. Further, when the equation of state is constant $w = p/\rho$, then the evolution of the energy density follows $\rho = \rho_0 a^{-3(1+w)}$. Plugging this into the Friedmann equation gives:

$$\dot{a}^2 = a^2 \frac{8\pi G}{3}\rho_0 a^{-3(1+w)} \tag{2}$$

Group the a factors and take a square root on both sides to get a differential equation for $a(t)$:

$$\begin{aligned}
\dot{a} &= \left(\frac{8\pi G}{3} a^{-3(1+w)+2}\right)^{1/2} \\
\frac{da}{dt} &= \sqrt{\frac{8\pi G}{3}} a^{\frac{-3(1+w)+2}{2}} \\
da &= \sqrt{\frac{8\pi G}{3}} a^{-(1+3w)/2} dt \\
a^{(1+3w)/2} da &= \sqrt{\frac{8\pi G}{3}} dt
\end{aligned} \tag{3}$$

We then integrate both sides:

$$\begin{aligned}
\int_0^a a^{(1+3w)/2} da &= \sqrt{\frac{8\pi G}{3}} \int_0^t dt \\
\frac{2a^{\frac{3(w+1)}{2}}}{3(w+1)} &= \sqrt{\frac{8\pi G}{3}} t
\end{aligned} \tag{4}$$

We can move the exponents of a to the right hand side, and we see that

$$a(t) \propto t^{\frac{2}{3(1+w)}} \tag{5}$$

b) If $w = -1$, then $\rho(a) = \rho_0$. That is, the right hand side of the Friedmann equation is constant, and we can instead write H_0^2 (more informative) than $8\pi G\rho_0/3$:

$$\begin{aligned}
\left(\frac{\dot{a}}{a}\right)^2 &= H_0^2 \\
\frac{\dot{a}}{a} &= H_0 \\
\frac{da}{da}\frac{1}{a} &= H_0 \\
\frac{da}{a} &= H_0 dt \\
\int \frac{da}{a} &= \int H_0 dt \\
\ln a &= H_0 t + C \\
a(t) &\propto \exp\left(H_0 t\right)
\end{aligned} \tag{6}$$

c) If $w = 0$ (that is, a matter dominated universe), then

$$a(t) \propto t^{2/3} \tag{7}$$

If $w = 1/3$ (that is, a radiation dominated universe), then

$$a(t) \propto t^{1/2} \tag{8}$$