Question 1

In the following, express the different ages as functions of the Hubble time, $1/H_0$.

a) Compute the age of a flat ($k = 0$) universe entirely dominated by matter ($\Omega_{\rm m} = 1$, all other density parameters are zero).

b) Compute the age of a flat ($k = 0$) Universe with $\Omega_m = 0.3$ and $\Omega_\Lambda = 0.7$. You can do the integral numerically if you want, although it also has an analytic solution (see hint below). Is this Universe younger or older than the purely matter-dominated universe of part (a)?

c) Compute the age of a curved matter-dominated universe with $\Omega_{\rm m} = 0.9$ and $\Omega_K = 0.1$? You can Taylor expand the integrand to perform the integral. Is this universe younger or older than a purely matter-dominated flat ($k = 0$) universe?

Hint: You may find this integral useful

$$\int_0^1 \frac{\sqrt{x}}{\sqrt{1+bx^3}}dx = \frac{2}{3\sqrt{b}}\sinh^{-1}(\sqrt{b}) \tag{1}$$

It is relevant to recall that all these integrals come from:

$$\begin{aligned} t_{\rm age} &= \int dt \\ &= \int \frac{da}{aH(a)} \end{aligned} \tag{2}$$

Where $H(a)$ depends on the contents of the universe:

$$H(a) = H_0\sqrt{\Omega_{\rm m}a^{-3} + \cdots} \tag{3}$$

a) There is a single component in this universe (matter), so

$$H(a) = H_0 a^{-3/2} \tag{4}$$

So the age of the universe is

$$\begin{aligned} t_{\rm age} &= \int_0^1 \frac{1}{aH_0a^{-3/2}}da \\ &= \frac{1}{H_0}\int_0^1 a^{1/2}da \\ &= \frac{1}{H_0}\cdot\frac{2a^{3/2}}{3}\Big|_0^1 \\ &= \frac{2}{3}H_0^{-1} \end{aligned} \tag{5}$$

b) In this scenario,

$$H(a) = H_0\sqrt{\Omega_\mathrm{m} a^{-3} + \Omega_\Lambda} \tag{6}$$

So

$$\begin{aligned}
t_\mathrm{age} &= H_0^{-1}\int_0^1 \frac{da}{a\sqrt{\Omega_\mathrm{m} a^{-3} + \Omega_\Lambda}} \\
&= H_0^{-1}\int_0^1 \frac{da}{a\sqrt{a^{-3}}\sqrt{\Omega_\mathrm{m} + \Omega_\Lambda a^3}} \\
&= H_0^{-1}\int_0^1 \frac{\sqrt{a}da}{\sqrt{\Omega_\mathrm{m} + \Omega_\Lambda a^3}} \\
&= H_0^{-1}\frac{1}{\sqrt{\Omega_\mathrm{m}}}\int_0^1 \frac{\sqrt{a}da}{\sqrt{1 + \frac{\Omega_\Lambda}{\Omega_\mathrm{m}} a^3}} \\
\text{where } b = \frac{\Omega_\Lambda}{\Omega_\mathrm{m}} = \frac{0.7}{0.3}; \qquad &= H_0^{-1}\frac{1}{\sqrt{\Omega_\mathrm{m}}}\frac{2}{3\sqrt{b}}\sinh^{-1}(\sqrt{b}) \\
&\approx 0.9640\ H_0^{-1}
\end{aligned} \tag{7}$$

This is 1.44 times older than the universe from part (a).

c) With

$$H(a) = H_0\sqrt{\Omega_\mathrm{m} a^{-3} + \Omega_K a^{-2}} \tag{8}$$

We have

$$\begin{aligned}
t_\mathrm{age} &= H_0^{-1}\int_0^1 \frac{da}{a\sqrt{\Omega_\mathrm{m} a^{-3} + \Omega_K a^{-2}}} \\
&= H_0^{-1}\int_0^1 \frac{da}{\sqrt{\Omega_\mathrm{m} a^{-1} + \Omega_K}} \\
\text{Using numerical integration} \qquad &\approx 0.6806\ H_0^{-1}
\end{aligned} \tag{9}$$

This is slightly older (1.0209 times older) than the universe from part (a).