Question 1

In this question, we consider a flat matter-dominated Universe ($\Omega_{\rm m} = 1$) with Hubble constant H_0.

a) Compute the angular diameter distance to redshift z.

b) Show that the angular diameter distance reaches a maximum. At what redshift $z_{\rm max}$ is this maximum achieved?

c) What does this imply for the angular size $\theta(z)$ of an object of fixed physical size l placed at redshift z? Sketch $\theta(z)$. What does this tell you about the physical meaning of the angular diameter distance?

a) Because we are considering a flat universe, it follows that $\Omega_{\rm K} = 0$, so d_A (the angular diameter distance) is given by

$$d_A = a\chi \tag{1}$$

In terms of redshift, $a = 1/(1+z)$ and[1]

$$\begin{aligned}\chi(z) &= \int_0^z \frac{dz'}{H(z')} \\ &= H_0^{-1}\int_0^z \frac{dz'}{(z+1)^{3/2}} \\ &= H_0^{-1} 2\left(1 - \frac{1}{\sqrt{1+z}}\right)\end{aligned} \tag{2}$$

So

$$\begin{aligned}d_A(z) &= \frac{1}{1+z} H_0^{-1} 2\left(1 - \frac{1}{\sqrt{1+z}}\right) \\ &= \frac{2\left(\sqrt{z+1} - 1\right)}{(z+1)^{3/2}}\ H_0^{-1}\end{aligned} \tag{3}$$

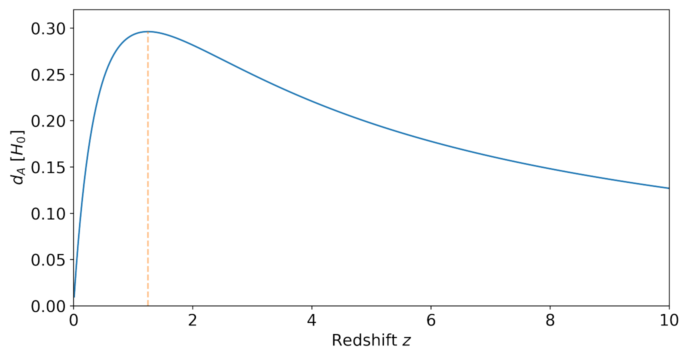
b) Find on the next page a plot of d_A. We can find $z_{\rm max}$ by differentiating $d_A(z)$, setting it equal to zero, and solving for z. Observe that

$$\begin{aligned}\frac{d(d_A)}{dz} &= \frac{1}{(z+1)^2} - \frac{3\left(\sqrt{z+1} - 1\right)}{(z+1)^{5/2}} \\ &= \frac{3}{(z+1)^{5/2}} - \frac{2}{(z+1)^2}\end{aligned} \tag{4}$$

[1] Since $\Omega_{\rm m} = 1$, the only contribution to the Friedmann equation is matter, hence $H(a) = H_0\sqrt{\cancel{\Omega_{\rm m}} a^{-3}} = H_0 a^{-3/2}$.

So

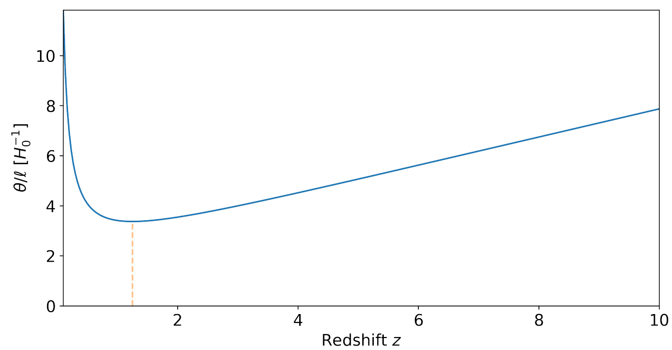
$$
\begin{aligned}
\frac{3}{(z_{\max}+1)^{5/2}} - \frac{2}{(z_{\max}+1)^2} &= 0 \\
(1+z_{\max})^{1/2} &= \frac{3}{2} \\
1+z_{\max} &= \frac{9}{4} \\
z_{\max} &= \frac{5}{4}
\end{aligned}
\tag{5}
$$



c) Recall that

$$\theta = \frac{l}{d_A} \tag{6}$$

Keep l fixed. Initially (for small values of z), increasing z implies increasing d_A which yields smaller angular sizes θ. This behavior stops when we hit the maximum value of d_A at $z_{\max}$, after which the angular diameter distance starts to decrease and θ starts to increase.



In the Euclidean sense (matching our everyday intuition), if I hold a pen in front of me and imagine a triangle between my nose and the endpoints of the pen, the angle at my nose becomes smaller as I move the pen farther away. In contrast, in our expanding universe, objects beyond a certain redshift $z_{\max}$ begin to appear larger with increasing distance. For reference, $z_{\max} = 5/4$ corresponds to a scale factor of $4/9$. This means that light from those redshifts (and higher) was emitted when the universe was less than half its current size. In that sense, we were physically closer to those objects at the time of emission, which explains why we perceive angular sizes θ that are larger than our Euclidean intuition would suggest.