ASTR 425/525
Cosmology

Homework Assignment 3

Due date: Wednesday October 22 2025, in class

Question 1 (8 points).

Consider the exact expression for the number density of a fermion of mass m in thermal equilibrium with zero chemical potential

$$n = g \int \frac{d^3p}{(2\pi)^3} \frac{1}{e^{\sqrt{p^2+m^2}/T} + 1}. \tag{1}$$

(a) Using $x = p/T$ and $y = m/T$, show that the above can be rewritten in the following form

$$\frac{n(y)}{T^3} = \frac{g}{2\pi^2} \int_0^\infty dx \, \frac{x^2}{e^{\sqrt{x^2+y^2}} + 1}. \tag{2}$$

(b) Using numerical integration and a plotting software of your choice, plot $n(y)/T^3$ as a function of $y = m/T$. Assume that $g = 2$ (like for an electron). Clearly label your axes.

(c) Now, add to your plot both the relativistic ($y \ll 1$) and non-relativistic ($y \gg 1$) limit for the number density of this fermion that we derived in class. Remember to convert them to functions of y, and plot $n(y)/T^3$ for those too. Confirm that these results match the exact result plotted in part (b) in the appropriate limit.

Question 2 (6 points).

When the chemical potential μ of a given species vanishes, the numbers of particles (n) and anti-particles ($\bar{n}$) of that species in the cosmic plasma are equal. However, the presence of $\mu \neq 0$ results in a net particle number density, $n - \bar{n} \neq 0$.

(a) For relativistic fermions with $\mu \neq 0$ and $T \gg m$, show that

$$\begin{aligned} n - \bar{n} &= \frac{g}{2\pi^2} \int_0^\infty dp\, p^2 \left(\frac{1}{e^{(p-\mu)/T} + 1} - \frac{1}{e^{(p+\mu)/T} + 1} \right) \\ &= \frac{g}{6\pi^2} T^3 \left[\pi^2 \left(\frac{\mu}{T} \right) + \left(\frac{\mu}{T} \right)^3 \right]. \end{aligned} \tag{3}$$

Note that this is an exact result in the relativistic limit and not a truncated series.

(b) Now, let's apply the above result to electrons and positrons in their relativistic limit ($T \gg 1$ MeV). Electric charge neutrality in the early Universe implies that $n_{\rm p} = n_{\rm e} - \bar{n}_{\rm e}$, where $n_{\rm p}$ is the number density of protons. Use this to show that, in the early Universe, we have

$$\frac{\mu_{\rm e}}{T} \simeq \frac{3\eta_{\rm b}\zeta(3)}{\pi^2}, \tag{4}$$

where $\eta_{\rm b} \equiv n_{\rm b}/n_\gamma$ is the baryon-to-photon ratio, with $n_{\rm b}$ and n_γ being the baryon and photon number densities, respectively. *Hint: relate the proton number density to the baryon number density. Do not forget the presence of neutrons. What is the relative abundance of neutrons and protons for $T \gg 1$ MeV?*

Question 3 (6 points).

As we discussed in class, cosmological neutrinos are slightly colder than cosmic microwave background photons today with

$$T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma. \tag{5}$$

where T_ν is the neutrino temperature, and T_γ is the photon temperature.

(a) Show that the combined number density of one generation of neutrinos and anti-neutrinos in the Universe today is

$$n_\nu = \frac{3}{11} n_\gamma, \tag{6}$$

where n_γ is the number density of photons today.

(b) Use the above result to show that the contribution from all species of non-relativistic massive neutrinos to the critical density of the Universe today is given by

$$\Omega_\nu h^2 = \frac{\sum_i m_{\nu,i}}{94\,\mathrm{eV}}, \tag{7}$$

where h is the reduced Hubble rate, and $\sum_i m_{\nu,i}$ denotes the sum of neutrinos masses, summed over all neutrino species.

Question 4 (8 points).

If the neutrinos were massless in our Universe, the relative energy density of neutrinos as compared to that of photons would be constant for $T \lesssim 0.1$ MeV (after e^+e^- annihilation), that is, $\rho_\nu/\rho_\gamma =$ constant.

(a) Assuming that all three neutrino species present in our Universe are massless, compute the ratio ρ_ν/ρ_γ for $T \lesssim 0.1$ MeV. Remember that $T_\nu = (4/11)^{1/3} T_\gamma$.

(b) In the real Universe, we know that at least two neutrino species have non-vanishing masses. A realistic model compatible with the results of neutrino oscillation experiments is to have one massless neutrino ($m_1 = 0$), a second one with mass $m_2 = 0.010$ eV, and a third one with mass $m_3 = 0.050$ eV. Compute the ratio $\rho_\nu(z)/\rho_\gamma(z)$ in this scenario over the redshift range $0 \le z \le 10^3$ and plot it using your favorite plotting package. Clearly label your axes. Add the constant line you found in part (a) to your plot as a comparison. Note that since massive neutrinos always decouple from the cosmic plasma while ultra-relativistic, their energy density (for one species) is given by

$$\rho_\nu(T_\nu) = g \int \frac{d^3p}{(2\pi)^3} \frac{\sqrt{p^2 + m_\nu^2}}{e^{p/T_\nu} + 1}. \tag{8}$$