ASTR 425/525
Cosmology

Homework Assignment 4
Due date: Monday November 10 2025, in class

Question 1 (6 points).

In class, we argued that the neutron freeze-out occurs at a temperature of $T_f \simeq 0.8$ MeV. This temperature can be estimated by comparing the weak interaction rate of the neutrons with the Hubble rate at that epoch. The interaction rate for the key reactions $p + \bar{\nu}_e \leftrightarrow n + e^+$ and $p + e^- \leftrightarrow n + \nu_e$ is given by

$$\Gamma_W(x) = \left(\frac{255}{\tau_n}\right)\frac{12 + 6x + x^2}{x^5}, \tag{1}$$

where $\tau_n = 886.7$ sec is the neutron lifetime, and $x = Q/T$, with $Q \equiv m_n - m_p = 1.2933$ MeV. On the other hand, the Hubble expansion rate can be gotten from the Friedmann equation

$$H^2 = \frac{8\pi G}{3}\rho_{\rm rad}, \qquad \text{with} \qquad \rho_{\rm rad} = \frac{\pi^2}{30}g_*(T)T^4. \tag{2}$$

Heuristically, neutron freeze-out will occur when $\Gamma_W \sim H$. A more precise value for T_f can be obtained by solving

$$\Gamma_W(T_f) = \frac{3}{2}H(T_f). \tag{3}$$

Using the expressions given above, show that $T_f \simeq 0.8$ MeV. Perhaps, the easiest way to do that is to plot both $\Gamma_W(T)$ and $H(T)$ and determine where they intersect. Be mindful of the units to make sure you are comparing the two rates in the same unit system. What value of $g_*(T)$ should you use in the above?

Question 2 (6 points).

In class, we mentioned several time that the age of the Universe was about 1 second when neutron froze out at $T_f = 0.8$ MeV. Let us derive this result. First, remember that the age of the Universe at scale factor a is given by

$$t(a) = \int_0^a \frac{da'}{a'H(a')}. \tag{4}$$

The issue is that we don't know H as a function of a at early times, but rather as a function of temperature

$$H(T) = \sqrt{\frac{8\pi G}{3}\rho_{\rm rad}(T)}, \qquad \text{with} \qquad \rho_{\rm rad} = \frac{\pi^2}{30}g_*(T)T^4. \tag{5}$$

To make matter worse, T does not scale as $1/a$ when g_* (or g_{*S}) is changing. However, we can derive an approximate expression that is pretty accurate by taking $g_*(T)$ to be constant and $T \propto 1/a$.

(a) Show that if $T \propto 1/a$, we have

$$\frac{dT}{T} = -\frac{da}{a}. \tag{6}$$

(b) Using the above and assuming $g_*(T)$ to be constant, show that the age of the Universe at temperature T was

$$\frac{t}{\text{sec}} \simeq \frac{2.42}{\sqrt{g_*}} \left(\frac{T}{\text{MeV}}\right)^{-2} . \tag{7}$$

(c) Using the appropriate value of g_* for $T \sim 1$ MeV, show that the age of the Universe at $T_f = 0.8$ MeV was $t \simeq 1.15$ sec.

Question 3 (3 points).

Use the Saha equation for the free electron fraction X_e,

$$\left(\frac{1 - X_e}{X_e^2}\right)_{\text{eq}} = \frac{2\zeta(3)}{\pi^2} \eta_{\text{b}} \left(\frac{2\pi T}{m_{\text{e}}}\right)^{3/2} e^{B_{\text{H}}/T}, \tag{8}$$

to show that the temperature of the Universe when 90% of the electrons have combined with protons to form neutral atoms (i.e. $X_e = 0.1$) is $T_{\text{rec}} \simeq 0.3$ eV. Why is $T_{\text{rec}} \ll B_{\text{H}} = 13.6$ eV?

Question 4 (3 points).

Even in the absence of recombination (that is, $X_e = n_{\text{e}}/(n_{\text{p}} + n_{\text{H}}) = 1$ at all times), the photons populating our Universe would eventually decouple from the baryons anyway due the volumetric dilution from the expansion. Estimate the temperature and redshift at which photon decoupling would occur in a Universe that is always ionized. Use $H_0 = 2.133h \times 10^{-33}$ eV, $h = 0.674$, $\Omega_{\text{m}} = 0.315$, $\eta_{\text{b}} = 6.1 \times 10^{-10}$, and $\sigma_{\text{T}} = 1.71 \times 10^{-3}$ MeV^{-2}.