ASTR 425/525
Cosmology

Homework Assignment 5
Due date: Monday November 24 2025, in class

Question 1 (5 points).

In class, we wrote down the decomposition on the celestial sphere of the cosmic microwave background (CMB) temperature anisotropies as

$$\frac{\delta T(\hat{\boldsymbol{n}})}{\bar{T}} \equiv \frac{T(\hat{\boldsymbol{n}}) - \bar{T}}{\bar{T}} = \sum_{l=2}^{\infty} \sum_{m=-1}^{l} a_{lm} Y_{lm}(\hat{\boldsymbol{n}}), \tag{1}$$

where $\hat{\boldsymbol{n}} = \hat{\boldsymbol{n}}(\theta, \phi)$ is a unit vector on the sphere, $\bar{T} = \langle T(\hat{\boldsymbol{n}}) \rangle$ is the average temperature of the CMB sky, and Y_{lm} are spherical harmonics, which form a complete basis for functions defined on the unit sphere. Here, $\langle \ldots \rangle$ denotes the ensemble average over different realizations of the CMB sky. We also have removed the dipole ($l = 1$) since it is dominated by our motion through space rather than the primary anisotropies.

(a) Given a measured map of $\delta T(\hat{\boldsymbol{n}})/\bar{T}$ on the sky, one can compute the corresponding a_{lm} coefficients. Use the orthogonality condition of spherical harmonics

$$\int d\Omega \, Y_{lm}(\hat{\boldsymbol{n}}) Y^*_{l'm'}(\hat{\boldsymbol{n}}) = \delta_{ll'} \delta_{mm'}, \tag{2}$$

where $d\Omega = \sin\theta d\theta d\phi$ is the solid angle integral measure, Y^*_{lm} is the complex conjugate of Y_{lm}, and δ_{ij} is the Kronecker delta, to show that the a_{lm} coefficients are given by

$$a_{lm} = \int d\Omega \, Y^*_{lm}(\hat{\boldsymbol{n}}) \, \frac{\delta T(\hat{\boldsymbol{n}})}{\bar{T}} \tag{3}$$

(b) Show that, by construction, the a_{lm} coefficients are zero-mean variables

$$\langle a_{lm} \rangle = 0. \tag{4}$$

(c) Therefore, the first nonzero statistical moment of the a_{lm} is their *variance* (two-point function), which can written as

$$\langle a_{lm} a^*_{l'm'} \rangle = C_l \delta_{ll'} \delta_{mm'}, \tag{5}$$

where the Kronecker deltas are a consequence of statistical isotropy. Here, C_l is referred to as the *angular temperature power spectrum*. The perhaps surprising thing in our Universe is that C_l captures the *entire* statistical properties of the CMB (that is, higher n-point functions are either zero or given by products of C_l). In other words, the a_{lm} are normally distributed random variables (i.e. a_{lm} are sampled from a Gaussian distribution with zero mean and standard deviation given by $\sqrt{C_l}$). Now, imagine that we have a full-sky, noise free, map of $\delta T(\hat{\boldsymbol{n}})/\bar{T}$. The a_{lm} for such a map can be computed using Eq. (3) above. For each l, the a_{lm}

could be seen as $2l+1$ independent samples from the Gaussian distribution with zero mean and variance C_l. From these measured a_{lm}, we can estimate the variance C_l as

$$\hat{C}_l = \frac{1}{2l+1}\sum_{m=-l}^{l}|a_{lm}|^2. \tag{6}$$

This estimator is unbiased, i.e. $\langle\hat{C}_l\rangle = C_l$, but it has a non-vanishing variance $\langle(C_l - \hat{C}_l)^2\rangle \neq 0$ between the true and estimated C_l, implying that even with a perfect noise-free map of the sky we cannot measure C_l to arbitrary precision. Show that any C_l measurement has an irreducible error given by

$$\frac{\Delta C_l}{C_l} \equiv \frac{\sqrt{\langle(C_l - \hat{C}_l)^2\rangle}}{C_l} = \sqrt{\frac{2}{2l+1}}. \tag{7}$$

This intrinsic error is usually referred to as *cosmic variance* and is worse for low values of l (large angular scales). Hint: use Wick's theorem to derive the above

$$\langle a_{lm}a^*_{lm}a_{lm'}a^*_{lm'}\rangle = \langle a_{lm}a^*_{lm}\rangle\langle a_{lm'}a^*_{lm'}\rangle + \langle a_{lm}a_{lm'}\rangle\langle a^*_{lm}a^*_{lm'}\rangle + \langle a_{lm}a^*_{lm'}\rangle\langle a^*_{lm}a_{lm'}\rangle. \tag{8}$$

Question 2 (10 points).

At the start of inflation, the typical size of the Universe (that is, the size of a causally connected region) was H_I^{-1}, where H_I was the Hubble expansion rate at that time. During inflation, this small causally connected region was stretched by a humongous factor

$$H_I^{-1} \to e^{H_I\Delta t}H_I^{-1}, \tag{9}$$

where Δt is the duration of inflation and H_I is constant. It is useful to define $N \equiv H_I\Delta t$, which represents the number of *e-folds* that occurred during inflation. The question we would like to answer now is how many e-folds of inflationary expansion are necessary to solve either the horizon or flatness problem.

(a) While the exact energy scale at which inflation occurred is unknown, a decent guess is that it happened at the Grand Unification scale at which the electroweak and strong force unify into a single interaction. It is believed that this transition occurs when the temperature of the Universe was $T_{\rm GUT} \simeq 10^{15}$ GeV. If this is the case, then the Hubble rate during inflation can be estimated from the Friedmann equation as

$$H_I^2 = \frac{8\pi G}{3}\frac{\pi^2}{30}g_*(T_{\rm GUT})T_{\rm GUT}^4. \tag{10}$$

Show that the Hubble rate during inflation in this scenario is $H_I \simeq 1.4\times10^{12}$ GeV. Assume only the particle content of the Standard Model. Using unit conversion, show that this corresponds to a Universe of approximate size $H_I^{-1} \simeq 1.4\times10^{-28}$ m at that time.

(b) Use the fact that $T_0 = 2.725$ K $= 2.348\times10^{-4}$ eV today to argue that the Universe has expanded by a factor of $\sim 4\times10^{27}$ between the end of inflation (when $T = 10^{15}$ GeV) and today. Assume that $T \propto 1/a$ always, where a is the scale factor.

(c) To solve the horizon problem, we need the initial causally connected region of size H_I^{-1} to be stretched such that the whole of the CMB last scattering surface today is causal. If the comoving radius of the last scattering surface today is $\sim 3.1 H_0^{-1}$, how many e-folds of inflation are necessary to make the last-scattering surface causal? Don't forget the amount of expansion that occurred between the end of inflation and today (i.e. the answer from part (b)). For this problem, it is useful to use the Hubble constant in natural units, $H_0 = 2.133h \times 10^{-33}$ eV, with $h = 0.674$.

(d) To solve the flatness problem, we need to explain why $|\Omega_K| < 0.001$. Writing

$$\Omega_K = \frac{\kappa}{R^2 H_0^2}, \tag{11}$$

where $\kappa = \{-1, 0, 1\}$ and R is the radius of curvature of the Universe, the constraint can be written as

$$R > 10^3 H_0^{-1}. \tag{12}$$

The flatness problem can be solved if the small causally connected region of size H_I^{-1} get stretched such that it has size $\geq R$ today. Using the above constraint on R, how many e-folds of inflation are necessary to solve the flatness problem. Again, do not forget to use your answer from part (b).