PHYS301
Homework 1 solutions

Spring 2026

Due: February 1st

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Problem 1

Prove Sterling's formula. Start by noting that factorials are related to the Gamma function via $N! = \Gamma(N+1)$, with

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx \ . \tag{1}$$

Use the above to write

$$N! = \int_0^\infty e^{-F(x)} dx \ . \tag{2}$$

After establishing what the function $F(x)$ is, find its minimum (call it x_0). Then, Taylor expand $F(x)$ around its minimum:

$$F(x) \approx F(x_0) + \frac{1}{2} F''(x_0)(x - x_0)^2 \ , \tag{3}$$

and use this in the above integral to establish that

$$N! \approx \sqrt{2\pi N} N^N e^{-N} \ , \tag{4}$$

when $N \gg 1$. How accurate is Stirling's formula for $N = 10$? What about $N = 100$?

Using the definition of the Gamma function, we have

$$\begin{aligned} N! &= \Gamma(N+1) \\ &= \int_0^\infty x^{(N+1)-1} e^{-x} dx \\ &= \int_0^\infty x^N e^{-x} dx \end{aligned} \tag{5}$$

To rewrite the integrand as an exponential, let's use the identity $x = e^{\ln x}$:

$$\begin{aligned} N! &= \int_0^\infty e^{\ln(x^N)} e^{-x} dx \\ &= \int_0^\infty e^{\ln(x^N) - x} dx \\ &= \int_0^\infty e^{N \ln(x) - x} dx \\ &= \int_0^\infty e^{-(x - N \ln(x))} dx \end{aligned} \tag{6}$$

So that

$$F(x) = x - N \ln(x) \tag{7}$$

To find the minimum, we can use the derivative $F'(x) = 1 - \frac{N}{x}$, which is 0 when $x = N$, hence

$$x_0 = N \tag{8}$$

To second order in x, the Taylor expansion of $F(x)$ about the minimum x_0 is

$$F(x) = (N - N \ln(N)) + \frac{(x - N)^2}{2N} + \mathcal{O}(x^3) \tag{9}$$

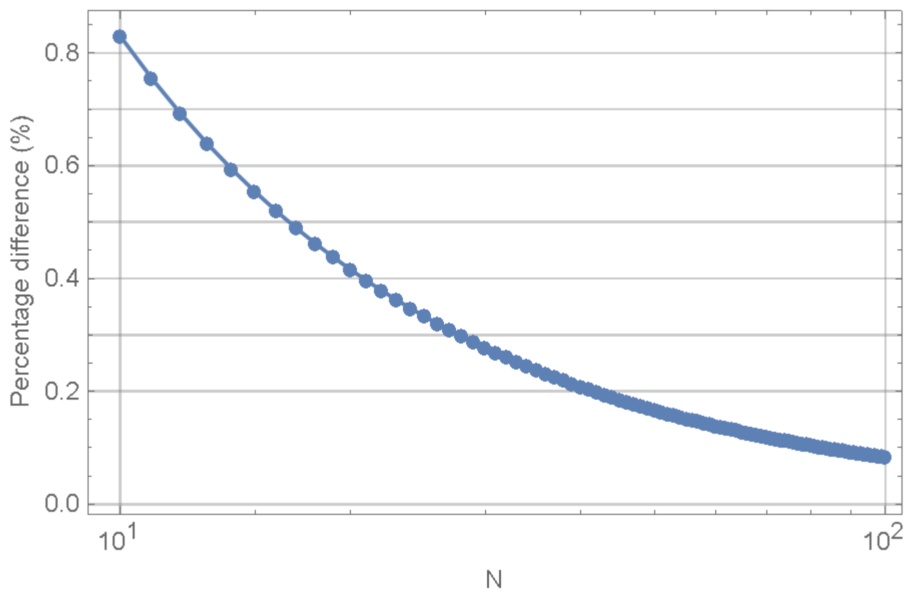
Where the second derivative $F''(x) = \frac{N}{x^2}$ was used. Using this in place of $F(x)$ in the integral yields

$$\begin{aligned} N! &\approx \int_0^\infty e^{-(N-N\ln(N))-\frac{(x-N)^2}{2N}}dx \\ &= \int_0^\infty e^{-N}e^{N\ln(N)}e^{-\frac{(x-N)^2}{2N}}dx \\ &= e^{-N}e^{\ln(N^N)}\int_0^\infty e^{-\frac{(x-N)^2}{2N}}dx \\ &= e^{-N}N^N\int_0^\infty e^{-\frac{(x-N)^2}{2N}}dx \\ &= e^{-N}N^N\sqrt{\frac{\pi N}{2}}\left(1+\text{Erf}(\sqrt{N/2})\right) \end{aligned} \tag{10}$$

In the limit $N \gg 1$, the error function (Erf) approaches 1, hence

$$N! \approx \sqrt{2\pi N}N^N e^{-N} \tag{11}$$

The larger N is, the better the approximation. We can see this behavior in the plot below.



At $N = 10$, Stirling's approximation is 0.82% smaller, while at $N = 100$ it is 0.083% smaller.

Problem 2

Let's consider a system of N non-interacting spins, which each can be in either the spin up or spin down state. Let $N_\uparrow$ be the number of spins in the up state, and $N_\downarrow = N - N_\uparrow$ the number of spins in the down state. Define the **spin excess** s as

$$2s \equiv N_\uparrow - N_\downarrow \, , \tag{12}$$

where the leading factor of 2 is just a convention.

a) Starting from the multiplicity for a macrostate with $N_\uparrow$ spins up,

$$\Omega(N, N_\uparrow) = \frac{N!}{N_\uparrow!(N - N_\uparrow)!} \, , \tag{13}$$

show that the multiplicity of a macrostate with spin excess s is

$$\Omega(N, s) = \frac{N!}{\left(\frac{1}{2}N + s\right)! \left(\frac{1}{2}N - s\right)!} \, . \tag{14}$$

b) In the limit that $s/N \ll 1$ and $N \gg 1$, show that this multiplicity is approximately Gaussian (up to normalization factor)

$$\Omega(N, s) \simeq \sqrt{\frac{2}{\pi N}} 2^N e^{-2s^2/N} \tag{15}$$

c) What is the standard deviation (width) of this Gaussian? Use this information to show that the width to height ration of the above multiplicity scales as

$$\sim \frac{N}{2^N} \tag{16}$$

for $N \gg 1$. Use this information to argue that the above multiplicity is *extremely* sharply peaked $s = 0$. If you were to draw in your homework the above $\Omega(N, s)$ function for $N = 1000$ with a height at $s = 0$ of 10 cm, what would be the width of the multiplicity that you would draw?

a) Using the fact that $N_\downarrow = N - N_\uparrow$, the spin excess can be written using N instead of $N_\downarrow$:

$$\begin{aligned} 2s &= N_\uparrow - N_\downarrow \\ &= N_\uparrow - (N - N_\uparrow) \\ &= 2N_\uparrow - N \end{aligned} \tag{17}$$

So

$$\begin{aligned} N_\uparrow &= \frac{2s + N}{2} \\ &= s + \frac{1}{2}N \end{aligned} \tag{18}$$

With this in mind, we can perform the direct substitution into $\Omega(N, N_\uparrow)$:

$$\begin{aligned}\Omega(N,s) &= \Omega(N,N_\uparrow)\Big|_{N_\uparrow = s+\frac{1}{2}N} \\ &= \frac{N!}{\left[s+\frac{1}{2}N\right]!\left(N-\left[s+\frac{1}{2}N\right]\right)!} \\ &= \frac{N!}{\left(\frac{1}{2}N+s\right)!\left(\frac{1}{2}N-s\right)!}\end{aligned} \tag{19}$$

b) Using Stirling's approximation,

$$N! \approx \sqrt{2\pi N}N^N e^{-N} , \tag{20}$$

we see that

$$\begin{aligned}\Omega(N,s) &\approx \frac{\sqrt{2\pi N}N^N e^{-N}}{\sqrt{2\pi\left(\frac{1}{2}N+s\right)}\left(\frac{1}{2}N+s\right)^{\left(\frac{1}{2}N+s\right)}e^{-\left(\frac{1}{2}N+s\right)}\sqrt{2\pi\left(\frac{1}{2}N-s\right)}\left(\frac{1}{2}N-s\right)^{\left(\frac{1}{2}N-s\right)}e^{-\left(\frac{1}{2}N-s\right)}} \\ &= \frac{\sqrt{2\pi N}N^N}{\sqrt{2\pi\left(\frac{1}{2}N+s\right)}\left(\frac{1}{2}N+s\right)^{\left(\frac{1}{2}N+s\right)}\sqrt{2\pi\left(\frac{1}{2}N-s\right)}\left(\frac{1}{2}N-s\right)^{\left(\frac{1}{2}N-s\right)}} \\ &\approx \sqrt{\frac{2\pi N}{2\pi\frac{1}{2}N\cdot 2\pi\frac{1}{2}N}}\cdot\frac{N^N}{\left(\frac{1}{2}N+s\right)^{\left(\frac{1}{2}N+s\right)}\left(\frac{1}{2}N-s\right)^{\left(\frac{1}{2}N-s\right)}} \\ &= \sqrt{\frac{2}{\pi N}}\cdot\frac{N^N}{\left(\frac{1}{2}N+s\right)^{\left(\frac{1}{2}N+s\right)}\left(\frac{1}{2}N-s\right)^{\left(\frac{1}{2}N-s\right)}} \\ &= \sqrt{\frac{2}{\pi N}}\cdot\frac{N^N}{N^{\left(\frac{1}{2}N+s\right)+\left(\frac{1}{2}N-s\right)}\left(\frac{1}{2}+\frac{s}{N}\right)^{\left(\frac{1}{2}N+S\right)}\left(\frac{1}{2}-\frac{s}{N}\right)^{\left(\frac{1}{2}N-s\right)}} \\ &= \sqrt{\frac{2}{\pi N}}\cdot\frac{1}{\left(\frac{1}{2}+\frac{s}{N}\right)^{\left(\frac{1}{2}N+S\right)}\left(\frac{1}{2}-\frac{s}{N}\right)^{\left(\frac{1}{2}N-s\right)}}\end{aligned} \tag{21}$$

To induce the desired exponential form from the remaining fraction, it's quite useful to the the logarithm of the entire expression. Once we have the logarithms in place, we can perform small-x approximation

$$\ln(1+x) \sim x - \frac{x^2}{2} + \mathcal{O}(x^3) \tag{22}$$

Observe then that

$$\begin{aligned}\ln\Omega &= \ln\left(\sqrt{\frac{2}{\pi N}}\right) - \left(\frac{N}{2}+s\right)\ln\left(\frac{1}{2}\left[1+\frac{2s}{N}\right]\right) - \left(\frac{N}{2}-s\right)\ln\left(\frac{1}{2}\left[1-\frac{2s}{N}\right]\right) \\ &\approx \ln\left(\sqrt{\frac{2}{\pi N}}\right) - \left(\frac{N}{2}+s\right)\left(\ln\frac{1}{2}+\frac{2s}{N}-\frac{2s^2}{N^2}\right) - \left(\frac{N}{2}-s\right)\left(\ln\frac{1}{2}-\frac{2s}{N}-\frac{2s^2}{N^2}\right) \\ &= \ln\left(\sqrt{\frac{2}{\pi N}}\right) - \frac{2s^2}{N} + N\ln(2)\end{aligned} \tag{23}$$

With the approximation performed, we can now take it back to Ω by exponentiating both sides

$$\begin{aligned}\Omega &= \sqrt{\frac{2}{\pi N}}\frac{e^{N\ln(2)}}{e^{2s^2/N}} \\ &= \sqrt{\frac{2}{\pi N}}\frac{2^N}{e^{2s^2/N}} \\ &\equiv \sqrt{\frac{2}{\pi N}}2^N e^{-2s^2/N}\end{aligned} \tag{24}$$

as expected.

c) We observe that Ω is a Gaussian over the variable s. By recalling the functional form of Gaussians, $\propto e^{-\frac{1}{2}\frac{s^2}{\sigma^2}}$, we see that

$$\sigma = \frac{\sqrt{N}}{2} \tag{25}$$

To find the width to height ratio, we first compute the height as the value at $s = 0$:

$$\text{height} = \Omega_{\text{max}} = \sqrt{\frac{2}{\pi N}}2^N \tag{26}$$

So the weight to height ratio is

$$\begin{aligned}\frac{\sigma}{\Omega_{\text{max}}} &= \frac{\sqrt{N}/2}{\sqrt{\frac{2}{\pi N}}2^N} \\ &= \frac{N}{2^N}\sqrt{\pi/8} \\ &\sim \frac{N}{2^N}\end{aligned} \tag{27}$$

Which implies that as N grows, this ratio shrinks to 0 extremely fast (linear vs exponential growth). As for the paper example, the with is extremely small:

$$\begin{aligned}\text{width} &= (\text{width to height ratio}) \cdot \text{height} \\ &= \frac{N}{2^N}\bigg|_{N=1000} \cdot 10 \text{ cm} \\ &= 9.33 \times 10^{-298} \text{ cm}\end{aligned} \tag{28}$$

In fact, unphysically small.

Problem 3

The meaning of "never." It has been said that "six monkeys, set to strum unintelligently on typewriters for millions of years, would be bound in time to write all the books in the British Museum." This statement is nonsense, for it gives a misleading conclusion about very, very large numbers. Could all the monkeys in the world have typed out a single specified book in the age of the universe?

Suppose that 10^{10} monkeys have been seated at typewriters throughout the age of the universe, 10^{18} seconds. This number of monkeys is about three times greater than the present human population of the earth. We suppose that a monkey can hit 10 typewriter keys per second. A typewriter may have 44 keys; we accept lowercase letters in place of capital letters. Assuming that Shakespeare's *Hamlet* has 10^5 characters, will the monkeys hit upon *Hamlet*?

a) Show that the probability that any given sequence of 10^5 characters typed at random will come out in the correct sequence (the sequence of *Hamlet*) is of the order of

$$\left(\frac{1}{44}\right)^{100{,}000} = 10^{-164{,}345}\ , \tag{29}$$

where we have used $\log_{10}44 = 1.64345$.

b) Show that the probability that a monkey-Hamlet will be typed in the age of the universe is approximately $10^{-164{,}316}$. The probability of *Hamlet* is therefore zero in any operational sense of an event, so that the original statement at the beginning of this problem is nonsense: one book, much less a library, will never occur in the total literary production of the monkeys.

a) There are $10^5 = 100,000$ spaces/slots to fill using a combination of 44 characters. Hamlet corresponds to the combination of the right key being pressed on the first space, then the right key on the second space, and so on until the $100,000$th space. The probability of randomly selecting the right key per space is 1/44, so the probability of doing so for all 100,000 spaces is

$$\begin{aligned}(1/44)^{100{,}000} &= 10^{\log_{10}((1/44)^{100{,}000})}\\ &= 10^{-100{,}000\cdot\log_{10}(44)}\\ &= 10^{-100{,}000\cdot 1.64345}\\ &= 10^{-164{,}345}\end{aligned} \tag{30}$$

b) If the monkeys can hit 10 keys per second, then it should take them $10^5/10 = 10^4$ seconds to type Hamlet (of course, assuming they are pressing the right keys). As such, they have plenty of time to try over and over again.

How many characters can they type in the time it takes for the universe to reach its current age? We see that

$$\begin{aligned}\text{Total keys pressed} &= (\text{Number of keys per second})\cdot(\text{Age of the universe (sec)})\cdot(\text{number of monkeys})\\ &= 10^{18}\cdot 10\cdot 10^{10}\\ &= 10^{29}\text{ keys}\end{aligned} \tag{31}$$

So in a long string of 10^{29} characters, what are the chances that Hamlet happens to be there?

$$10^{29} \times 10^{-164,345} = 10^{-164,316}$$