PHYS301
Homework 2 solutions

Spring 2026

Due: February 8th

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Problem 1

We saw in class that the multiplicity of an ideal gas with N particles, volume V, and total energy E is

$$\Omega(N,V,E) = \sigma(N,V)E^{3N/2} \ , \tag{1}$$

where $\sigma(N,V)$ is independent of E.

a) Show that the relationship between total energy and temperature is

$$E = \frac{3}{2}Nk_BT \ . \tag{2}$$

b) Compute the heat capacity of the system.

c) Derive an expression for the change in entropy of this system as we heat the system from temperature T_1 to T_2.

a) We know the multiplicity Ω, so we can write the entropy of the system:

$$\begin{aligned} S(E) &\equiv k_B \ln(\Omega) \\ &= k_B \ln(\sigma(N,V)E^{3N/2}) \\ &= k_B \ln(\sigma(N,V)) + k_B \ln(E^{3N/2}) \\ &= k_B \ln(\sigma(N,V)) + k_B \frac{3N}{2}\ln(E) \end{aligned} \tag{3}$$

And with an expression of entropy S, we can compute the inverse temperature $1/T$ of the system:

$$\begin{aligned} \frac{1}{T} &\equiv \frac{\partial S}{\partial E} \\ &= k_B \frac{3N}{2}\frac{\partial}{\partial E}\ln(E) \\ &= k_B \frac{3N}{2}\frac{1}{E} \end{aligned} \tag{4}$$

Where the term $k_B \ln(\sigma(N,V))$ vanished as it has no E dependence. We can rewrite this to solve for E:

$$E = \frac{3}{2}Nk_BT \ , \tag{5}$$

as expected.

b) The answer from part (a) allows us to write the entropy S of the system as a function of temperature T rather than energy:

$$S(T) = k_B \ln(\sigma(N,V)) + k_B \frac{3N}{2}\ln(\frac{3}{2}Nk_BT) \tag{6}$$

And with S as a function of T, we can compute the heat capacity using its definition:

$$\begin{aligned} \frac{C}{T} &= \frac{\partial S}{\partial T} \\ &= k_B \frac{3N}{2}\frac{1}{T} \end{aligned} \tag{7}$$

so that

$$C = k_B \frac{3N}{2} \tag{8}$$

That is, a heat capacity that is independent of temperature.

c) Recall that the change in entropy is given by the integral of $C(T)/T$, so:

$$\begin{aligned}
\Delta S &= \int_{T_1}^{T_2} \frac{C(T)}{T} dT \\
&= \int_{T_1}^{T_2} k_B \frac{3N}{2} \frac{1}{T} dT \\
&= k_B \frac{3N}{2} \Big(\ln(T_2) - \ln(T_1) \Big) \\
&= k_B \frac{3N}{2} \ln\left(\frac{T_2}{T_1}\right)
\end{aligned} \tag{9}$$

Problem 2

Consider an Einstein solid with N oscillators, each with natural frequency ω, sharing q quanta of energy. Neglecting the zero-point energy of the oscillators, the total energy of the system is

$$E_{\text{tot}} = q\hbar\omega \; . \tag{10}$$

a) Using the multiplicity of microstates for an Einstein solid, compute the entropy $S(N, q)$ of this system in the $N \gg 1$ limit, using Stirling's formula to simplify your answer.

b) By writing the entropy as a function of N and E_{tot}, $S(N, E_{\text{tot}})$, derive an expression for E_{tot} as a function of the temperature T.

c) At which value of E_{tot} (and T) is the entropy maximized? Does this value make sense to you? Explain why.

d) Explain physically the behavior of the system as $T \to 0$ and $T \to \infty$. Assuming $E_{\text{tot}} > 0$, can an Einstein solid have negative temperature?

a) Recall that entropy S was defined (in relation to the multiplicity Ω of the system) as

$$S = k_B \ln \Omega \tag{11}$$

For the Einstein solid, the multiplicity with N oscillators and q total quanta is

$$\Omega(N, q) = \frac{(q + N - 1)!}{q!(N-1)!} \equiv \binom{q + N - 1}{q} \tag{12}$$

In the $N \gg 1$ limit, the above reduces to

$$\Omega \overset{N \gg 1}{=} \frac{(q + N)!}{q!(N)!} \tag{13}$$

Further, we can write the factorials using Stirling's approximation, $N! \approx \sqrt{2\pi N} N^N e^{-N}$:

$$\begin{aligned} \Omega &\approx \frac{\sqrt{2\pi(q+N)}(q+N)^{(q+N)} e^{-(q+N)}}{\sqrt{2\pi q} q^q e^{-q} \sqrt{2\pi N} N^N e^{-N}} \\ &= \frac{1}{\sqrt{2\pi}} q^{-\frac{1}{2}-q} N^{-\frac{1}{2}-N} (q+N)^{\frac{1}{2}+q+N} \end{aligned} \tag{14}$$

So the entropy of the system is

$$\begin{aligned} & S = k_B \ln(\Omega) \\ N \gg 1 \text{ \& Stirling's...} \quad & \approx k_B \ln\left(\frac{1}{\sqrt{2\pi}} q^{-\frac{1}{2}-q} N^{-\frac{1}{2}-N} (q+N)^{\frac{1}{2}+q+N}\right) \\ & = k_B \left[\ln\frac{1}{\sqrt{2\pi}} - \left(\frac{1}{2}+q\right)\ln(q) - \left(\frac{1}{2}+N\right)\ln(N) + \left(\frac{1}{2}+q+N\right)\ln(q+N)\right] \\ N \gg 1 \ldots \quad & = k_B \left(-q\ln(q) - N\ln N + (q+N)\ln(q+N)\right) \end{aligned} \tag{15}$$

b) Although we could substitute $q \to E_{\text{tot}}/\hbar\omega$, to avoid carrying many constants, let's instead use chain rule. By definition, $1/T$ is

$$\begin{aligned}\frac{1}{T} &= \frac{\partial S}{\partial E_{\text{tot}}} \\ &= \frac{\partial q}{\partial E_{\text{tot}}}\frac{\partial S}{\partial q} \\ &= \frac{1}{\hbar\omega}\frac{\partial S}{\partial q} \\ &= \frac{1}{\hbar\omega}k_B\Big(\ln(q+N) - \ln(q)\Big) \\ &= \frac{1}{\hbar\omega}k_B \ln\frac{q+N}{q} \\ &= \frac{1}{\hbar\omega}k_B \ln\left(1+\frac{N}{q}\right) \\ &= \frac{1}{\hbar\omega}k_B \ln\left(1+\frac{N\hbar\omega}{E_{\text{tot}}}\right)\end{aligned} \tag{16}$$

It is now straightforward to solve for E_{tot} as a function of T:

$$\begin{aligned}\frac{\hbar\omega}{k_BT} &= \ln\left(1+N\frac{\hbar\omega}{E_{\text{tot}}}\right) \\ e^{\hbar\omega/k_BT} &= 1+N\frac{\hbar\omega}{E_{\text{tot}}} \\ e^{\hbar\omega/k_BT} - 1 &= N\frac{\hbar\omega}{E_{\text{tot}}} \\ E_{\text{tot}}(T) &= \frac{N\hbar\omega}{e^{\hbar\omega/k_BT}-1}\end{aligned} \tag{17}$$

c) A maximum occurs when the derivative of $S(E_{\text{tot}})$ is zero, but this quantity is (by definition) $1/T$, so maximum entropy occurs when $1/T \to 0$, that is: when $T \to \infty$. In the limit of high T, $e^{\hbar\omega/k_BT} \approx 1 + \frac{\hbar\omega}{k_BT}$, so:

$$\begin{aligned}E_{\text{tot}}(T) &\overset{\text{high } T}{=} \frac{N\hbar\omega}{1+\frac{\hbar\omega}{k_BT}-1} \\ &= \frac{N\hbar\omega}{\hbar\omega/k_BT} \\ &= Nk_BT\end{aligned} \tag{18}$$

So entropy is maximized when $E \to \infty$. From a microstates point of view, recall that entropy increases with the number of microstates (Ω). If we increase the energy E_{tot}, we increase q, which itself increases Ω hence a larger entropy. There is no maximum entropy in this system. The energy levels have no ceiling to reach.

d) As $T \to 0$, the exponential in the denominator of E_{tot} dominates, sending $E_{\text{tot}} \to 0$. This corresponds to having most (if not all) oscillators in their ground state.

As $T \to \infty$, $E \to Nk_BT \to \infty$, which is the case we discussed in part (c). In contrast to the "every oscillator exists in the ground state" picture of the $T \to 0$ limit, observe how here we can say that, on average, each oscillator has energy k_BT (hence if we have N oscillators, the total energy is $\sim Nk_BT$).

As for negative temperatures, the Einstein solid can't have them. To see this, recall again that if we increase the energy of the system, the entropy of the Einstein solid always increases. This positive rate of change is what defines temperature, so temperature is always positive.

Problem 3

Consider a system of N interacting spins. At low temperatures, the interactions ensure that all spins are either aligned or anti-aligned with the z axis, even in the absence of an external magnetic field.

At high temperatures, the interactions become less important and spins can point in either $\pm z$ direction. If the heat capacity takes the form

$$C = C_{\text{max}} \left(\frac{2T}{T_0} - 1 \right) \tag{19}$$

for $T_0/2 < T < T_0$ and it is $C = 0$ otherwise. Determine C_{max}.

Recall that

$$\frac{\partial S}{\partial T} = \frac{C(T)}{T} \tag{20}$$

Further, for a system of N interacting spins, entropy does attain a maximum (unlike the system discussed in problem 2), making any change in entropy a finite (bounded) quantity. In fact, at high temperatures the entropy of the system is

$$\begin{aligned} S &= k_B \ln(\Omega_{\text{high-}T}) \\ &= k_B \ln(2^N) \\ &= k_B N \ln(2) \end{aligned} \tag{21}$$

Where $\Omega = 2^N$ comes from the fact that each one of the N spins can point either in the positive or negative z direction.

Integrating[1] eq. 20 using the expression for $C(T)$ as given in Eq. 19 yields a change in entropy

$$\begin{aligned} \Delta S &= \int_{T_0/2}^{T_0} \frac{C(T)}{T} dt \\ &= C_{\text{max}} \left(1 - \ln(2)\right) \end{aligned} \tag{22}$$

From which we can say that

$$C_{\text{max}} = \frac{\Delta S}{1 - \ln(2)} \tag{23}$$

By noting that entropy is $S = k_B \ln(1) = 0$ at $T = 0$ (all spins pointing down), it is clear that $\Delta S = k_B N \ln(2)$, so:

$$C_{\text{max}} = \frac{k_B N \ln(2)}{1 - \ln(2)} \tag{24}$$

[1] Observe that the T integral goes from 0 to ∞, but the piecewise form of $C(T)$ vanishes all but the segment from $T = T_0/2$ to $T = T_0$.