PHYS301
Homework 3 solutions

Spring 2026

Due: February 15th

Contents

Problem 1 [4 points]

Consider a system consisting of N independent spin 1/2 particles, each of which can be in one of two quantum states: 'up' and 'down'. In a magnetic field B, the energy of a spin in the up state is $-\mu B/2$ and that of the down state is $+\mu B/2$, where μ is the magnetic moment of a particle.

a) Show that the partition function is

$$Z = 2^N \cosh^N \left(\frac{\mu B}{2k_B T} \right) \tag{1}$$

b) Find the average energy E and entropy S. Check that your results for both quantities make sense for $T = 0$ and $T = \infty$.

c) Compute the magnetization M of the system as a function of temperature, defined by

$$M \equiv \mu \langle N_\uparrow - N_\downarrow \rangle \tag{2}$$

where $N_\uparrow$ and $N_\downarrow$ are the number of up and down spins, respectively.

d) The magentic susceptibility is defined as $\chi \equiv \frac{\partial M}{\partial B}$. Derive Curie's law which states that at high temperatures

$$\chi(T) \propto \frac{1}{T} \tag{3}$$

a) To write the partition function Z of the system, let us first find the partition function for one of the (observe that they are **independent**!) particles. By definition, Z is a sum over the possible states of the system $Z = \sum_n e^{-\beta E_n}$. A given particle can be in one of two states: spin up or down, so

$$\begin{aligned} Z_{\text{Single particle}} &= e^{-\beta E_\uparrow} + e^{-\beta E_\downarrow} \\ &= e^{\beta \mu B/2} + e^{-\beta \mu B/2} \\ &= 2 \frac{e^{\beta \mu B/2} + e^{-\beta \mu B/2}}{2} \\ &= 2 \cosh \frac{\beta \mu B}{2} \\ &= 2 \cosh \frac{\mu B}{2k_B T} \end{aligned} \tag{4}$$

Where I used the fact that

$$\cosh x = \frac{e^x + e^{-x}}{2} \tag{5}$$

If we have N of these independent particles, then the partition function of the system Z is the product of the individual partition functions, hence

$$Z = 2^N \cosh^N \left(\frac{\mu B}{2k_B T} \right) \tag{6}$$

b) Let's start with the average density:

$$\begin{aligned}\langle E\rangle &= -\frac{\partial}{\partial\beta}\ln(Z)\\ &= -\frac{\partial}{\partial\beta}\ln\left(2^N\cosh^N\left(\frac{\mu B}{2k_BT}\right)\right)\\ &= -N\frac{\partial}{\partial\beta}\ln\left(\cosh\left(\frac{\beta\mu B}{2}\right)\right)\\ &= -N\frac{\mu B}{2}\tanh\left(\frac{\mu B\beta}{2}\right)\end{aligned} \tag{7}$$

As $T\to 0$, $\beta\to\infty$, and as $\beta\to\infty$, the hyperbolic tangent goes to 1. This gives an average energy of $-N\mu B/2$, which is corresponds to having all of the particles with their spin up, the state which lowest energy.

Analogously, when $T\to\infty$, $\beta\to 0$, and as $\beta\to 0$, the hyperbolic tangent goes to 0. This gives an average energy of 0, which can be understood as the limit in which the magnetic field can't align spins anymore, so on average you have as many spins pointing up as you have pointing down, hence the 0 average energy.

Let's now consider entropy. By direct computation:

$$\begin{aligned}S &= k_B\frac{\partial}{\partial T}(T\ln Z)\\ &= k_B\frac{\partial}{\partial T}\left[T\ln\left(2^N\cosh^N\left(\frac{\mu B}{2k_BT}\right)\right)\right]\\ &= k_B\ln\left(2^N\cosh^N\left(\frac{B\mu}{2k_BT}\right)\right)-\frac{B\mu N\tanh\left(\frac{B\mu}{2k_BT}\right)}{2T}\\ &= Nk_B\ln(2)+Nk_B\ln\left(\cosh\left(\frac{B\mu}{2k_BT}\right)\right)-\frac{NB\mu\tanh\left(\frac{B\mu}{2k_BT}\right)}{2T}\\ &= N\left[k_B\ln(2)+k_B\ln\left(\cosh\left(\frac{B\mu}{2k_BT}\right)\right)-\frac{B\mu\tanh\left(\frac{B\mu}{2k_BT}\right)}{2T}\right]\end{aligned} \tag{8}$$

As $T\to 0$, $S\to 0$ (effectively $\infty-\infty=0$), so the system's entropy is lowest at $T=0$.

As $T\to\infty$, the hyperbolic tangent term in the expression for entropy goes to 0, but the $\ln(\cdot)$ term attains a finite limit: In this limit, the argument inside the hyperbolic cosine goes to 0, and $\cosh(0)=1$, so $S\to k_B\ln(2^N)$. This matches the definition $S=k_B\ln(\Omega)$, with $\Omega=2^N$, and it makes physical sense: at extremely high temperatures the magnetic field won't be able to align spins.

Further, this complements the answer to the $T\to 0$ limit. The state of lowest energy has 1 configuration, so $\Omega=1$, yielding an entropy of $S=k_B\ln(1)=0$, as discussed before.

c) We know that the *exact* energy of the system is given by

$$\begin{aligned} E &= N_\uparrow E_\uparrow + N_\downarrow E_\downarrow \\ &= N_\uparrow\left(-\frac{\mu B}{2}\right) + N_\downarrow\left(+\frac{\mu B}{2}\right) \\ &= -\frac{\mu B}{2}(N_\uparrow - N_\downarrow) \end{aligned} \tag{9}$$

Where $N_\uparrow$ ($N_\downarrow$) is the number of particles in the system with their spin up (down). Taking the average of both sides, we find that $\langle N_\uparrow - N_\downarrow\rangle$ is related to $\langle E\rangle$, a quantity we found in part (b):

$$\begin{aligned} \langle -\frac{\mu B}{2}(N_\uparrow - N_\downarrow)\rangle &= \langle E\rangle \\ \langle N_\uparrow - N_\downarrow\rangle &= -\frac{2}{\mu B}\langle E\rangle \\ \text{(using the result form part (b))} \qquad &= -\frac{2}{\mu B}\left(-N\frac{\mu B}{2}\tanh\left(\frac{\mu B\beta}{2}\right)\right) \\ &= N\tanh(\beta\mu B/2) \end{aligned} \tag{10}$$

With this in mind, we can now write the magnetization M of the system directly:

$$\begin{aligned} M &= \mu\langle N_\uparrow - N_\downarrow\rangle \\ &= \mu N\tanh(\beta\mu B/2) \end{aligned} \tag{11}$$

d) From part (c) we know that $M = \mu\tanh^N(\beta\mu B/2)$. By definition of magnetic susceptibility, we have

$$\begin{aligned} \chi &= \frac{\partial M}{\partial B} \\ &= \mu N\frac{\partial}{\partial B}\left(\tanh(\beta\mu B/2)\right) \\ &= \frac{1}{2}\beta\mu^2 N\operatorname{sech}^2\left(\frac{B\beta\mu}{2}\right) \end{aligned} \tag{12}$$

Taking the high temperature limit, $T\to\infty$, so that $\beta\to 0$, we see that the hyperbolic secant goes to 1, so at high temperatures the magnetic susceptibility is

$$\chi \overset{T\to\infty}{=} \frac{1}{2}\beta\mu^2 N \propto \beta \propto \frac{1}{T} \tag{13}$$

as expected. ■

Problem 2 [3 points]

A quantum violin string can vibrate at frequencies

$$\omega,\ 2\omega,\ 3\omega \text{ and so on...}$$

Each vibration mode can be treated as an independent quantum harmonic oscillator.

Ignore the zero point energy, so that the mode with frequency $p\omega$ has energy $E = n\hbar p\omega$, for $n \in \mathbb{Z}$. What is the average energy of the string at temperature T?

A (quantum) violin string can vibrate at difference frequencies (quantized by ω). Each one of the frequencies can have a different energy level. Formally, a string vibrating at frequency ωp has energy $n\hbar p\omega$ when it vibrates in its nth energy level. Do note that ignoring the zero point energy means shifting the energy spectrum, while keeping the $n = 0$ state (now with 0 energy).

With this in mind, we have an infinite number of quantum harmonic oscillators, each corresponding to one of the frequencies ω, 2ω, etc. These oscillators can be treated as independent, so it suffices to derive an expression for the partition function for one of these oscillators (with frequency $p\omega$):

$$\begin{aligned} Z_p &= \sum_{n=0}^{\infty} e^{-\beta E_n} \\ &= \sum_{n=0}^{\infty} e^{-\beta n\hbar p\omega} \\ \text{(Let } x_p = e^{-\beta\hbar p\omega}\text{)} \qquad &= \sum_{n=0}^{\infty} x_p^n \\ &= \frac{1}{1 - x_p} \\ &= \frac{1}{1 - e^{-\beta\hbar p\omega}} \end{aligned} \tag{14}$$

The *total* partition function for the system is then the product of the individual partition functions:

$$\begin{aligned} Z &= \prod_{p=1}^{\infty} Z_p \\ &= \prod_{p=1}^{\infty} \frac{1}{1 - e^{-\beta\hbar p\omega}} \end{aligned} \tag{15}$$

The average energy of the system follows directly:

$$\begin{aligned}
\langle E\rangle &= -\frac{\partial}{\partial\beta}\ln Z \\
&= -\frac{\partial}{\partial\beta}\ln\left(\prod_{p=1}^{\infty}\frac{1}{1-e^{-\beta\hbar p\omega}}\right) \\
&= \sum_{p=1}^{\infty}-\frac{\partial}{\partial\beta}\ln\left(\frac{1}{1-e^{-\beta\hbar p\omega}}\right) \\
&= \sum_{p=1}^{\infty}\langle E\rangle_p
\end{aligned} \tag{16}$$

Where the infinite product got pulled out as a sum because of the sum-product log property: $\ln(xy) = \ln(x)+\ln(y)$. From this we see that the average energy of the system is simply the sum of the average energies of each one of the harmonic oscillators. Computing one of them individually yields:

$$\begin{aligned}
\langle E\rangle_p &= -\frac{\partial}{\partial\beta}\ln Z_p \\
&= \frac{p\omega\hbar}{e^{p\beta\omega\hbar}-1}\ ,
\end{aligned} \tag{17}$$

allowing us to write the average energy of the whole system as a sum of them:

$$\begin{aligned}
\langle E\rangle &= \sum_{p=1}^{\infty}\langle E\rangle_p \\
&= \sum_{p=1}^{\infty}\frac{p\omega\hbar}{e^{p\beta\omega\hbar}-1}
\end{aligned} \tag{18}$$

■

Problem 3 [3 points]

Compute the entropy of a one-dimensional ideal gas with N particles of mass m at temperature T, confined to a line of length L.

The generalization of the partition function to (1 dimensional) continuous systems is

$$Z = \frac{1}{h}\int dx dp e^{-\beta H(x,p)} \tag{19}$$

Where h is Planck's constant and H is the Hamiltonian. Because we are considering an ideal gas, the Hamiltonian only contains kinetic energy (i.e., no interactions), so that

$$H(x,p) = \frac{p^2}{2m} \tag{20}$$

With this in mind, the partition function for **single** ideal gas particle confined to 1 dimension (0 to L) is

$$\begin{aligned} Z_1 &= \frac{1}{h}\int_{x=0}^{L}\int_{p=-\infty}^{\infty} e^{-\beta p^2/2m} dp dx \\ &= \frac{L}{h}\int_{-\infty}^{\infty} e^{-\beta p^2/2m} dp \\ &= \frac{L}{h}\sqrt{\frac{2\pi m}{\beta}} \\ &= \frac{L}{h}\sqrt{2\pi m k_B T} \end{aligned}$$

If we have N particles, then the partition function for the system is

$$\begin{aligned} Z &= \frac{Z_1^N}{N!} \\ &= \frac{L^N}{N! h^N}\left(2\pi m k_B T\right)^{N/2} \end{aligned} \tag{21}$$

Where the factor of $N!$ accounts for the multiplicity of the system (these particles are indistinguishable). The entropy of the system then follows by direct computation:

$$\begin{aligned} S &= k_B \frac{\partial}{\partial T}(T \ln Z) \\ &= k_B \frac{\partial}{\partial T}\left(T \ln\left(\frac{L^N}{N! h^N}\left(2\pi m k_B T\right)^{N/2}\right)\right) \\ &= k_B \frac{N}{2} + k_B \ln\left(\frac{L^N (2\pi k_B m T)^{N/2}}{N! h^N}\right) \end{aligned} \tag{22}$$

Let's introduce the thermal de Broglie wavelength

$$\lambda(T) \equiv \frac{h}{\sqrt{2\pi m k_B T}} \tag{23}$$

for which we get:

$$\begin{aligned} S &= k_B \frac{N}{2} + k_B \ln \left(\frac{L^N \sqrt{2\pi k_B m T}^N}{N! h^N} \right) \\ &= k_B \frac{N}{2} + k_B \ln \left(\frac{L^N (h/\lambda)^N}{N! h^N} \right) \\ &= k_B \frac{N}{2} + k_B \ln \left(\frac{L^N}{N! \lambda^N} \right) \\ &= k_B \frac{N}{2} + k_B N \ln(L/\lambda) - k_B \ln(N!) \end{aligned} \tag{24}$$

It is sensible that in a gas the value of N is large, so we can employ Stirling's[1] approximation to rewrite the last term:

$$\begin{aligned} S &= k_B \frac{N}{2} + k_B N \ln(L/\lambda) - k_B \left(N \ln N - N \right) \\ &= N k_B \left(\frac{1}{2} + \ln \frac{L}{\lambda} - \ln(N) + 1 \right) \\ &= N_B \left(\ln \frac{L}{\lambda N} + \frac{3}{2} \right) \end{aligned} \tag{25}$$

This now resembles the Sackur-Tetrode formula (for the entropy of a monatomic ideal gas), here in 1 dimension. ∎

[1] $\ln(x!) = x \ln x - x \mathcal{O}(\ln x)$