

PHYS301  
Homework 5 solutions

Spring 2026

Due: March 8th

**Contents**

Problem 1 [10 points]	2
Problem 2 [3 points]	7
Problem 3 [4 points]	8

## Problem 1 [10 points]

You probably know that atmospheric pressure on Earth decreases as we go up in altitude. We can understand this behavior by modeling the atmosphere as different layers of monoatomic ideal gas that are diffusive (chemical) and thermal equilibrium. Consider one such layer at a fixed height  $h$  above the surface of the Earth. It is made of an ideal gas at temperature  $T$ , with  $N$  atoms of mass  $m$  and volume  $V$ . Here,  $N$  will be a function of the height, that is,  $N = N(h)$ .

- a) By first computing the partition function  $Z$ , show that the Helmholtz free energy  $F = -k_B T \ln Z$  for this atmospheric layer is

$$F = -k_B T \ln \left( \frac{V^N}{N! \lambda_Q^3} \right) + mghN \quad (1)$$

where  $g$  is the gravitational acceleration, and  $\lambda_Q$  is the usual Thermal de Broglie wavelength.

- b) From the free energy, show that the chemical potential of an atmospheric layer at a fixed height  $h$  above the surface of the Earth is

$$\mu(h) = k_B T \ln (n(h) \lambda_Q^3) + mgh \quad (2)$$

where  $n(h) \equiv N(h)/V$  is the number density of atoms at height  $h$ .

- c) Use the fact that the different atmospheric layers are in diffusive equilibrium

$$\mu(h_1) = \mu(h_2)$$

to determine how  $n(h)$  varies with altitude. Assume the temperature to be the same across all atmospheric layers (this is usually not true on Earth). It is easiest to choose  $h_1 = 0$  and  $h_2 = h$ , and express your answer in terms of  $n(0)$ , the number density of gas at the Earth's surface.

- d) Use the ideal gas law to derive the pressure profile  $P(h)$  of our atmosphere. Assuming that our atmosphere is made of nitrogen ( $N_2$ ,  $m = 48 \times 10^{-27}$  kg) at  $T = 227$  Kelvin, plot  $P(h)/P(0)$  as a function of  $h$ . What is  $P(h)/P(0)$  at the top of mount Everest ( $h \simeq 8,850$  meters)? What about at the altitude at which modern planes fly ( $h \simeq 11,300$  meters)?

- a) The Hamiltonian  $H$  for each particle is

$$\begin{aligned} H_{\text{one particle}} &= \text{kinetic} + \text{potential} \\ &= \frac{p^2}{2m} + mgh \end{aligned} \quad (3)$$

This generalizes to  $N$  particles as a sum over the system. Assuming that we will look at a slice of the atmosphere at a time (i.e., a fixed height  $h$ ) the total potential energy of the system is  $Nmgh$ , where  $N$  is the number of particles:

$$H = \sum_{i=1}^N \frac{p_i^2}{2m} + Nmgh \quad (4)$$

Where  $p_i$  is the momentum of the  $i$ th particle. With  $\beta = 1/k_B T$ , the partition function is given by

$$Z = \frac{1}{N! h_{\text{Pl}}^{3N}} \int d^{3N} x d^{3N} p \exp(-\beta H) \quad (5)$$

Where  $h_{\text{Pl}}$  is to be understood as Planck's constant. The unusual labeling here prevents us from mixing it with the height  $h$ .<sup>1</sup> It is important here to observe that we have a 3-dimensional integral in space and in momentum per particle, hence the powers of  $3N$ . The integrals factor out nicely:

$$\begin{aligned} Z &= \frac{1}{N! h_{\text{Pl}}^{3N}} \int e^{-\beta \sum_i^N p_i^2/2m} e^{-\beta mgh} d^{3N} x d^{3N} p \\ &= \frac{e^{-\beta N mgh}}{N! h_{\text{Pl}}^{3N}} \int e^{-\beta \sum_i^N p_i^2/2m} d^{3N} x d^{3N} p \\ &= \frac{e^{-\beta N mgh}}{N! h_{\text{Pl}}^{3N}} \left( \int d^{3N} x \right) \left( \int e^{-\beta \sum_i^N p_i^2/2m} d^{3N} p \right) \\ &= \frac{e^{-\beta N mgh}}{N! h_{\text{Pl}}^{3N}} V^N \left( \int e^{-\beta \sum_i^N p_i^2/2m} d^{3N} p \right) \\ &= \frac{e^{-\beta N mgh}}{N! h_{\text{Pl}}^{3N}} V^N \left( \int e^{-\beta p^2/2m} d^3 p \right)^N \\ &= \frac{e^{-\beta N mgh}}{N! h_{\text{Pl}}^{3N}} V^N \left( 4\pi \int e^{-\beta p^2/2m} p^2 dp \right)^N \\ &= \frac{e^{-\beta N mgh}}{N! h_{\text{Pl}}^{3N}} V^N \left( 4\pi \sqrt{\frac{\pi}{2}} \sqrt{m^3/\beta^3} \right)^N \\ &= \frac{e^{-\beta N mgh}}{N! h_{\text{Pl}}^{3N}} V^N \left( \sqrt{2^3 \pi^3 m^3/\beta^3} \right)^N \\ &= \frac{e^{-\beta N mgh}}{N! h_{\text{Pl}}^{3N}} V^N \left( \frac{2\pi m}{\beta} \right)^{(3N/2)} \end{aligned} \quad (6)$$

By introducing the thermal de Broglie wavelength  $\lambda_Q$ :

$$\lambda_Q = \frac{h}{\sqrt{2\pi m k_B T}} \quad (7)$$

We see that

$$Z = \frac{V^N}{N! \lambda_Q^{3N}} e^{-\beta N mgh} \quad (8)$$

With an expression for the partition function  $Z$ , we can now compute the (Helmholtz) free

---

<sup>1</sup>Once we introduce the de Broglie wavelength, Planck's constant is absorbed away. As such, it is more sensible to rename this instead of  $h$ , a variable which we will keep for the rest of the problem.

energy directly:

$$\begin{aligned}
 F &= -k_B T \ln Z \\
 &= -k_B T \ln \left( \frac{V^N}{N! \lambda_Q^{3N}} e^{-\beta N m g h} \right) \\
 &= -k_B T \ln \left( \frac{V^N}{N! \lambda_Q^{3N}} \right) - k_B T \ln (e^{-\beta N m g h}) \\
 &= -k_B T \ln \left( \frac{V^N}{N! \lambda_Q^{3N}} \right) + k_B T \beta N m g h \\
 &= -k_B T \ln \left( \frac{V^N}{N! \lambda_Q^{3N}} \right) + N m g h
 \end{aligned} \tag{9}$$

b) Recall that

$$\mu = \left. \frac{\partial F}{\partial N} \right|_{T, V} \tag{10}$$

Before computing the derivative, consider applying Stirling's approximation on  $N!$ :

$$\begin{aligned}
 F &= -k_B T \ln \left( \frac{V^N}{N! \lambda_Q^{3N}} \right) + N m g h \\
 &= -k_B T \ln \left( \frac{V^N}{\lambda_Q^{3N}} \right) + k_B T \ln(N!) + N m g h \\
 &= -k_B T \ln \left( \frac{V^N}{\lambda_Q^{3N}} \right) + k_B T (N \ln N - N) + N m g h \\
 &= -k_B T \ln \left( \frac{V^N}{N^N \lambda_Q^{3N}} \right) - N k_B T + N m g h \\
 &= -k_B T N \ln \left( \frac{V}{N \lambda_Q^3} \right) - N k_B T + N m g h
 \end{aligned} \tag{11}$$

So  $\mu$  is

$$\begin{aligned}
 \mu(h) &= \frac{\partial}{\partial N} \left( -k_B T N \ln \left( \frac{V}{N \lambda_Q^3} \right) - N k_B T + N m g h \right) \\
 &= -k_B T + m g h - k_B T \frac{\partial}{\partial N} \left( N \ln \left( \frac{V}{N \lambda_Q^3} \right) \right) \\
 &= -k_B T + m g h - k_B T (-1 + \ln(V/\lambda_Q^3 N)) \\
 &= -k_B T \ln(1/n(h) \lambda_Q^3) + m g h \\
 &= k_B T \ln(n(h) \lambda_Q^3) + m g h
 \end{aligned} \tag{12}$$

where  $n(h) = N(h)/V$ .

c) By definition of diffusive equilibrium, we have

$$\mu(h_1) = \mu(h_2) \quad (13)$$

Let  $h_1 \equiv h$  and  $h_2 = 0$ . Then

$$\begin{aligned} \mu(h) &= \mu(0) \\ k_B T \ln(n(h)\lambda_Q^3) + mgh &= k_B T \ln(n(0)\lambda_Q^3) \\ \ln(n(h)\lambda_Q^3) &= \frac{k_B T \ln(n(0)\lambda_Q^3) - mgh}{k_B T} \\ n(h)\lambda_Q^3 &= \exp\left(\frac{k_B T \ln(n(0)\lambda_Q^3)}{k_B T}\right) \exp\left(-\frac{mgh}{k_B T}\right) \\ n(h) &= n(0) \exp\left(-\frac{mgh}{k_B T}\right) \end{aligned} \quad (14)$$

A result we can explain right away: as height increases, the density falls exponentially.

d) Recall that for an ideal gas (with  $N$  particles), we have

$$PV = Nk_B T \quad (15)$$

Which we can write as

$$\begin{aligned} P(h) &= \frac{N}{V} k_B T \\ &= n(h) k_B T \end{aligned} \quad (16)$$

Since

$$P(0) = n(0) k_B T \quad (17)$$

It follows that

$$\begin{aligned} \frac{P(h)}{P(0)} &= \frac{n(h)}{n(0)} \\ &= \exp\left(-\frac{mgh}{k_B T}\right) \end{aligned} \quad (18)$$

Find a plot on the next page.<sup>2</sup>

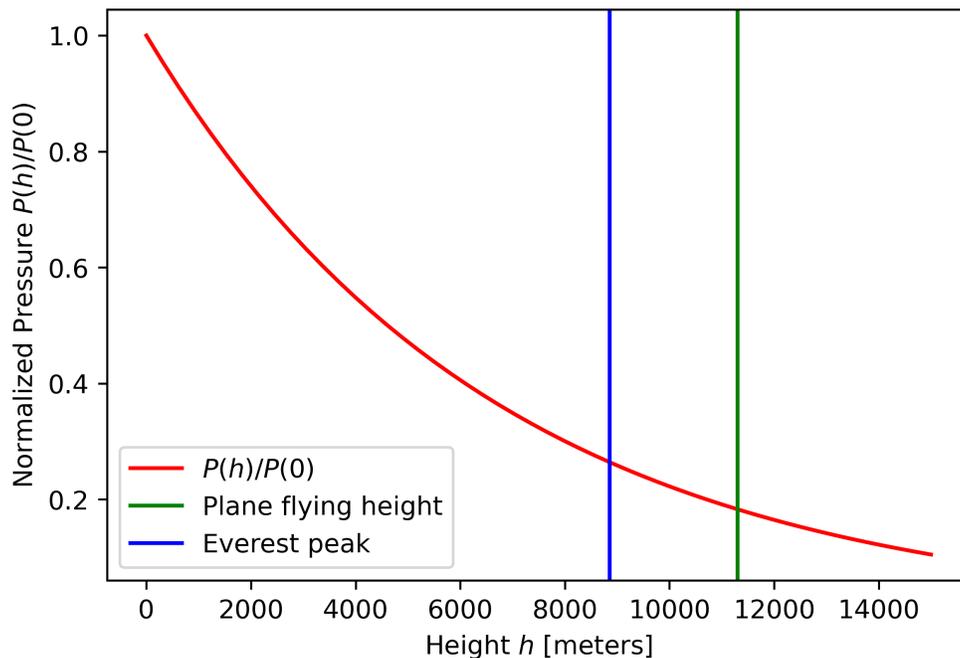
$$\begin{aligned} P(h_{\text{Everest}})/P(0) &= 0.264399 \\ P(h_{\text{Modern planes}})/P(0) &= 0.182945 \end{aligned} \quad (19)$$

<sup>2</sup>It is simplest to plot in meters, with  $g = 9.81 \text{ m/s}^2$ . By sticking to SI units, we are able to write Boltzmann's constant as  $k_B = 1.38 \times 10^{-23} \text{ J} \cdot \text{K}^{-1}$ , and we keep the provided temperature in Kelvin of course.

```

1 import numpy as np
2 import matplotlib.pyplot as plt
3
4 # Constants
5 g = 9.81 # gravity, m/s^2
6 kB = 1.38e-23 # Boltzmann, J/K
7
8 # Model parameters
9 mass_nitrogen = 48e-27 # kg
10 T = 227 # temperature, Kelvin
11
12 everest = 8850 # meters
13 planes = 11300 # meters
14
15 def normalized_pressure(h):
16     return np.exp(- mass_nitrogen * g * h / (kB*T))
17
18 heights = np.linspace(0,15000)
19 pressures = normalized_pressure(heights)
20
21 plt.plot(heights, pressures, color="red", label=r"$P(h)/P(0)$")
22 plt.axvline(x=planes, color="green", label="Plane flying height")
23 plt.axvline(x=everest, color="blue", label="Everest peak")
24 plt.xlabel(r"Height $h$ [meters]")
25 plt.ylabel(r"Normalized Pressure $P(h)/P(0)$")
26 plt.legend(loc="lower left")
27 plt.savefig("Problem_1_part_D.png", dpi=450, bbox_inches='tight')
28
29 print(f"Everest: {normalized_pressure(everest):.6f}")
30 print(f"Planes: {normalized_pressure(planes):.6f}")

```



## Problem 2 [3 points]

A neutral gas consists of  $N_e$  electrons  $e^-$ ,  $N_p$  protons  $p^+$  and  $N_H$  Hydrogen atoms  $H$ .

An electron and proton can combine to form Hydrogen



From the free energy of the system  $F(T, V; N_e, N_p, N_H)$ , we can define a chemical potential for each of the three species as

$$\mu_i = \frac{\partial F}{\partial N_i} \quad (20)$$

where  $i = e, p, H$ . By minimizing the free energy at fixed temperature and volume (i.e., set  $dF = 0$  with  $dT = dV = 0$ ), show that the condition for equilibrium is

$$\mu_e + \mu_p = \mu_H \quad (21)$$

Hint: How are  $dN_e$ ,  $dN_p$ , and  $dN_H$  related to each other? You don't need to derive an explicit expression for the free energy to do this problem.

Recall

$$dF = -SdT - PdV + \sum_i \mu_i dN_i \quad (22)$$

If  $dT = dV = 0$ , then

$$dF = \mu_e dN_e + \mu_p dN_p + \mu_H dN_H \quad (23)$$

Given the presented process, it is clear that by creating one hydrogen we lose one electron and one proton (from their isolated counts). This suggests

$$dN_H = -dN_e = -dN_p \quad (24)$$

Then

$$dF = (\mu_H - \mu_e - \mu_p) dN_H \quad (25)$$

If we want  $dF = 0$ , then we must have  $(\mu_H - \mu_e - \mu_p) = 0$ , so

$$\mu_e + \mu_p = \mu_H \quad (26)$$

■

### Problem 3 [4 points]

Consider a two-state system of a ground state with energy 0 and an excited state with energy  $\epsilon$ .

- a) Assuming that the system can be occupied by at most one particle, show that the grand canonical partition function (Gibbs sum) takes the form

$$\mathcal{Z} = 1 + \lambda + \lambda e^{-\beta\epsilon} \quad (27)$$

where  $\lambda = e^{\beta\mu}$ , with  $\mu$  being the chemical potential and  $\beta = 1/(k_B T)$ .

- b) Derive an expression for the average occupancy  $\langle n \rangle$  of the system.  
 c) Derive an expression for the average energy of the system.  
 d) Now assume that the system can host at most two particles, show that the grand canonical partition function takes the form

$$\begin{aligned} \mathcal{Z} &= 1 + \lambda + \lambda e^{-\beta\epsilon} + \lambda^2 e^{-2\beta\epsilon} \\ &= (1 + \lambda)(1 + \lambda e^{-\beta\epsilon}) \end{aligned} \quad (28)$$

What does the above factorization of the partition function tell you about the system?

- a) Recall that

$$\mathcal{Z} = \sum_n e^{-\beta(E_n - \mu N_n)} \quad (29)$$

Where the index  $n$  will run over 3 possible states for the system:

- (1) There is no particle in the system.

Then  $E_n = 0$  and  $N_n = 0$ . So

$$\begin{aligned} e^{-\beta(E_n - \mu N_n)} &= e^0 \\ &= 1 \end{aligned} \quad (30)$$

- (2) There is a particle, and it is in the ground state.

Then  $E_n = 0$  and  $N_n = 1$ . So

$$\begin{aligned} e^{-\beta(E_n - \mu N_n)} &= e^{\beta\mu} \\ &= \lambda \end{aligned} \quad (31)$$

Where  $\lambda$  is defined as  $\lambda = e^{\beta\mu}$ .

- (3) There is a particle, and it is in the excited state.

Then  $E_n = \epsilon$  and  $N_n = 1$ . So

$$\begin{aligned} e^{-\beta(E_n - \mu N_n)} &= e^{-\beta(\epsilon - \mu)} \\ &= e^{-\beta\epsilon} e^{\beta\mu} \\ &= \lambda e^{-\beta\epsilon} \end{aligned} \quad (32)$$

Adding these 3 states together, we get

$$\mathcal{Z} = 1 + \lambda + \lambda e^{-\beta\epsilon} \quad (33)$$

as expected.

b) By direct computation:

$$\begin{aligned} \langle N \rangle &= \frac{1}{\beta} \frac{\partial}{\partial \mu} \ln \mathcal{Z} \\ &= \frac{e^{\beta\mu}(1 + e^{\beta\epsilon})}{e^{\beta\epsilon} + e^{\beta\mu} + e^{\beta\epsilon}e^{\beta\mu}} \\ &= \frac{\lambda(1 + e^{\beta\epsilon})}{e^{\beta\epsilon} + \lambda + e^{\beta\epsilon}\lambda} \end{aligned} \quad (34)$$

c) By direct computation:

$$\begin{aligned} \langle E \rangle &= \mu \langle N \rangle - \frac{\partial}{\partial \beta} \ln \mathcal{Z} \\ &= \mu \frac{\lambda(1 + e^{\beta\epsilon})}{e^{\beta\epsilon} + \lambda + e^{\beta\epsilon}\lambda} - \frac{e^{\beta\mu}\mu + e^{-\beta\epsilon}e^{\beta\mu}(\mu - \epsilon)}{1 + e^{\beta\mu} + e^{-\beta\epsilon}e^{\beta\mu}} \\ &= \mu \frac{\lambda(1 + e^{\beta\epsilon})}{e^{\beta\epsilon} + \lambda + e^{\beta\epsilon}\lambda} - \frac{\lambda\mu + e^{-\beta\epsilon}\lambda(\mu - \epsilon)}{1 + \lambda + e^{-\beta\epsilon}\lambda} \\ &= \frac{\lambda\epsilon}{e^{\beta\epsilon} + \lambda + e^{\beta\epsilon}\lambda} \end{aligned} \quad (35)$$

d) Although the system hosts 2 particles, at most one particle can occupy a given state. As we did in part (a), let's go over the possible states:

(1) No particles in the system.

Then as in part (a),  $e^{-\beta(E_n - \mu N_n)} = 1$ .

(2) Only 1 particle in the system. Let it be in the ground state.

Then as in part (a),  $e^{-\beta(E_n - \mu N_n)} = \lambda$ .

(3) No particles in the system. Let it be in the excited state.

Then as in part (a),  $e^{-\beta(E_n - \mu N_n)} = \lambda e^{-\beta\epsilon}$ .

(4) Two particles in the system.

This implies  $N_n = 2$ .

Further, they must occupy the ground state and the excited state separately, hence total energy  $E_n = \epsilon + 0 = \epsilon$ .

We see then that

$$\begin{aligned} e^{-\beta(E_n - \mu N_n)} &= e^{2\beta\mu} e^{-\beta\epsilon} \\ &= (e^{\beta\mu})^2 e^{-\beta\epsilon} \\ &= \lambda^2 e^{-\beta\epsilon} \end{aligned} \quad (36)$$

Combining these 4 possibilities, we see that

$$\mathcal{Z} = 1 + \lambda + \lambda e^{-\beta\epsilon} + \lambda^2 e^{-2\beta\epsilon} \quad (37)$$

Which can be written as (factorized)

$$\mathcal{Z} = (1 + \lambda)(1 + \lambda e^{-\beta\epsilon}) \quad (38)$$

Which can be read as the system factorized into two sub-systems: one where a particle can exist in the ground state and another subsystem in which a particle can exist in the excited state. Further, it shows that these systems are independent from each other (no interactions).

$$\mathcal{Z} = \mathcal{Z}_{\text{ground state}} \mathcal{Z}_{\text{excited state}} \quad (39)$$

■