PHYS301
Homework 7 solutions

Spring 2026

Due: March 30th

Contents

Problem 1 [10 points]

In class, we saw that the partition function at temperature T for photons of a given (angular) frequency was given by

$$\begin{aligned} Z_\omega &= \sum_{N=0}^{\infty} e^{-\beta N \hbar\omega} \\ &= \frac{1}{1-e^{-\beta\hbar\omega}} \end{aligned} \tag{1}$$

Since photons don't interact with one another, the different photon frequency modes are independent of each other. The total partition function Z is thus simply the product of the partition function of different frequency modes ω_i:

$$Z = \prod_i Z_{\omega_i} \tag{2}$$

a) Use the above to compute the logarithm of the total partition function for a photon gas at temperature T in a box of volume V and show that it is equal to

$$\ln(Z) = -\frac{V}{\pi^2 c^3}\int_0^\infty d\omega\, \omega^2 \ln\left(1-e^{-\hbar\omega/k_BT}\right) \tag{3}$$

You will need the fact that the number of photon states in an infinitesimal phase-space volume $d^3x d^3p$ is

$$2\frac{d^3x d^3p}{h^3} \tag{4}$$

together with the fact that the energy of a photon is $E = \hbar\omega = pc$, where p is the momentum.

b) Using the Helmholtz free-energy

$$F = -k_BT\ln Z \tag{5}$$

compute the pressure and entropy of this photon gas. In particular, show that the pressure of the photon gas in the box is $p = \rho/3$, where ρ is the average energy density (energy per unit volume) in the box.

a) My taking the log of equation 2, the product turns into a sum:

$$\begin{aligned} \ln Z &= \ln\left(\prod_i Z_{\omega_i}\right) \\ &= \sum_i \ln(Z_{\omega_i}) \\ &= \sum_i \ln\left(\frac{1}{1-e^{-\beta\hbar\omega_i}}\right) \\ &= -\sum_i \ln(1-e^{-\beta\hbar\omega_i}) \end{aligned} \tag{6}$$

Using the infinitesimal phase-space volume for photons, we can turn the discrete sum into a (continuous) integral

$$\begin{aligned}\sum_i &= \int 2\frac{d^3x d^3p}{h^3} \\ &= 2\frac{V}{h^3}\int d^3p\end{aligned} \tag{7}$$

Where the space integral (d^3x) was taken over some finite volume V. Recall that in spherical coordinates the volume element is

$$d^3p = 4\pi p^2 dp \tag{8}$$

(with angular part integrated out). Using the energy relation for photons, observe that $p = \hbar\omega/c$, hence

$$\begin{aligned} dp &= \frac{\hbar}{c}d\omega \\ \Rightarrow \quad p^2 dp &= \frac{\hbar^3\omega^2}{c^3}d\omega \\ &= \frac{h^3\omega^2}{(2\pi)^3c^3}d\omega \end{aligned} \tag{9}$$

hence

$$\ln(Z) = -\frac{V}{\pi^2c^3}\int_0^\infty d\omega\ \omega^2 \ln\left(1 - e^{-\hbar\omega/k_BT}\right) \tag{10}$$

b) The integral in $\ln(Z)$ converges:

$$\int_0^\infty d\omega\ \omega^2 \ln\left(1 - e^{-\hbar\omega/k_BT}\right) = -\frac{\pi^4}{45\beta^3\hbar^3} \tag{11}$$

Hence

$$\begin{aligned} F &= -k_BT\ln Z \\ &= -k_BT\left(-\frac{V}{\pi^2c^3}\right)\left(\int_0^\infty d\omega\ \omega^2\ln\left(1-e^{-\hbar\omega/k_BT}\right)\right) \\ &= -k_BT\left(-\frac{V}{\pi^2c^3}\right)\left(-\frac{\pi^4}{45\beta^3\hbar^3}\right) \\ &= -k_BTV\frac{\pi^2}{45c^3\beta^3\hbar^3} \\ &= -V\frac{\pi^2}{45c^3\beta^4\hbar^3} \end{aligned} \tag{12}$$

This allows us to compute the pressure and entropy of the system:

$$p = -\left(\frac{\partial F}{\partial V}\right)_T \qquad S = -\left(\frac{\partial F}{\partial T}\right)_V \tag{13}$$

$$= \frac{\pi^2}{45c^3\beta^4\hbar^3} \qquad = k_BV\frac{\pi^2}{45c^3\hbar^3}k_B^3(4T^3) \tag{14}$$

$$= 4k_BV\frac{\pi^2}{45c^3\beta^3\hbar^3} \tag{15}$$

Lastly, the energy density is given by

$$
\begin{aligned}
\rho &= \frac{U}{V} \\
&= -\frac{1}{V}\frac{\partial\beta}{\partial}\ln(Z) \\
&= -\frac{1}{V}\frac{\partial}{\partial\beta}\left(-\frac{V}{\pi^2c^3}\right)\left(-\frac{\pi^4}{45\beta^3\hbar^3}\right) \\
&= \frac{V}{V}\frac{\pi^2}{45c^3\beta^4\hbar^3} \\
&= 3\frac{\pi^2}{45c^3\beta^4\hbar^3} \\
&= 3p
\end{aligned} \tag{16}
$$

Hence

$$\frac{p}{\rho} = \frac{1}{3}$$

■

Problem 2 [10 points]

The purpose of this question is to explain why the cosmic microwave background (CMB) radiation still has a black body spectrum, even though it has not been in thermal equilibrium with matter since very early in the universe's history. Consider a region of volume V in the cosmos containing black body radiation of temperature T. Suppose the cosmos expands (slowly) by a scale factor $a > 1$ (such that the volume increases such that $V \to a^3 V$), so that the momentum p and angular frequency ω of each electromagnetic radiation mode are rescaled by $1/a$.

a) If there are no electrically charged particles around to emit or absorb photons, the total number of photons in the expanded volume is the same as the initial one. Use this fact to show that the temperature in the expanded universe is lower by a factor of $1/a$. Thus, the CMB cools as the Universe expands.

b) Use this scaling of the temperature to show that the Planck function remains valid after the expansion (that is, it maintains its shape), up to an overall scaling factor. From this scaling factor, show that the energy density (energy per unit volume) is down by a factor of $1/a^4$ after the expansion.

c) Using the expression for the entropy of a photon gas derived in question 1, show that the entropy in the expanded volume is the same as the original entropy, thus confirming that the expansion of the Universe does not create entropy (in other words, it is adiabatic).

a) With the phase-space volume for photons from problem 1, we have that the number of photons in some volume V is

$$\begin{aligned} N &= \int_0^\infty \frac{V\omega^2}{\pi^2 c^3} \frac{1}{e^{\hbar\omega/k_B T} - 1} d\omega \\ (x = \hbar\omega/k_B T) \qquad &= \frac{V}{\pi^2} c^3 \left(\frac{k_B T}{\hbar}\right)^3 \int_0^\infty \frac{x^2}{e^x - 1} dx \\ &= \frac{V}{\pi^2} c^3 \left(\frac{k_B T}{\hbar}\right)^3 2\zeta(3) \\ &\propto VT^3 \end{aligned} \tag{17}$$

If total number of photons is conserved, then (where a prime denotes "after expansion"):

$$\begin{aligned} N' &= N \\ V'T'^3 &= VT^3 \\ a^3 V T'^3 &= VT^3 \\ T' &= \frac{T}{a} \end{aligned} \tag{18}$$

as expected.

b) Recall the Planck function:

$$u(\omega, T) = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{e^{\hbar\omega/k_B T} - 1} \tag{19}$$

After some expansion in the universe, $T \to T/a = T'$ and $\omega \to \omega/a = \omega'$, and the Planck function becomes

$$\begin{aligned} u(\omega', T') &= u(\omega/a, T/a) \\ &= \frac{\hbar}{\pi^2 c^3} \frac{\omega^3/a^3}{e^{\hbar\omega a/k_B T a} - 1} \\ &= \frac{1}{a^3} \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{e^{\hbar\omega/k_B T} - 1} \\ &= \frac{1}{a^3} u(\omega, T) \end{aligned} \tag{20}$$

So the expansion of the universe only rescales the Planck function.

Recall that the energy density is computed as

$$\rho = \frac{U}{V} = \int_0^\infty u(\omega, T) d\omega \tag{21}$$

(i.e., an integral of the Planck function). We already saw that after expansion, the Planck function scales as $1/a^3$. Similarly, $\omega \to \omega/a$, hence $d\omega \to d\omega/a$. With this in mind, the energy density after some expansion is

$$\begin{aligned} \rho' &= \int_0^\infty u(\omega', T') d\omega' \\ &= \int_0^\infty \frac{u(\omega, T)}{a^3} \frac{d\omega}{a} \\ &= \frac{1}{a^4} \int_0^\infty u(\omega, T) d\omega \\ &= \frac{1}{a^4} \rho \end{aligned} \tag{22}$$

Whence ρ scales as $1/a^4$ under expansion.

c) Recall from problem 1 that

$$S = 4k_B V \frac{\pi^2}{45 c^3 \beta^3 \hbar^3} \tag{23}$$

The entropy after expansion, call it S', is given by the above equation evaluated at

$$\begin{aligned} T' &= T/a \\ V' &= V a^3 \end{aligned}$$

The expansion relation for T translates to a relation for β as

$$\beta' = a\beta \tag{24}$$

Hence

$$\begin{aligned} S' &= 4k_B V' \frac{\pi^2}{45c^3 \beta'^3 \hbar^3} \\ &= 4k_B V \cancel{\alpha^3} \frac{\pi^2}{45c^3 \beta^3 \cancel{\alpha^3} \hbar^3} \\ &= 4k_B V \frac{\pi^2}{45c^3 \beta^3 \hbar^3} \\ &= S \end{aligned} \tag{25}$$

So we conclude that entropy is conserved.

$\blacksquare$