PHYS301
Homework 8 solutions

Spring 2026

Due: April 6th

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Problem 1 [4 points]

One of the first experimental realizations of a Bose-Einstein condensate used Rubidium-87 atoms cooled to extremely low temperatures. Imagine that we have 10^4 Rubidium-87 atoms confined to a tiny volume $V = 10^{-15}$ m^3.

a) Compute the critical temperature T_C below which a Bose-Einstein condensate starts to form.

b) Suppose that $T = 0.9T_C$. How many atoms are in the ground state? How about at $T = 0.1T_C$?

a) With

$$\begin{aligned} N &= 10^4 && (1)\\ V &= 10^{-15}\ \mathrm{m}^3 && (2)\\ m_{\mathrm{Rb}} &= 1.442 \times 10^{-25}\ \mathrm{kg} && (3)\\ k_B &= 1.3806 \times 10^{-23}\ \mathrm{J \cdot K^{-1}} && (4)\\ h &= 6.626 \times 10^{-34}\ \mathrm{J \cdot s} && (5)\\ \zeta(3/2) &= 2.61237 && (6) \end{aligned}$$

We use the expression for T_C derived in class:

$$\begin{aligned} T_C &= \frac{1}{\pi m k_B}\left(\frac{Nh^3}{V\zeta(3/2)}\right)^{2/3} \\ &= 1.71776 \times 10^{-7}\ \text{Kelvin} \\ &= 0.1717\ \mu\mathrm{K} \end{aligned} \tag{7}$$

This number might seem small, but we have achieved condensates at even lower temperatures (in the order of $T \sim 10^{-12}$ Kelvin!)

b) Recalling that

$$N_0 = N\left(1 - \left[\frac{T}{T_C}\right]^{3/2}\right) \tag{8}$$

We see that

$$\begin{aligned} &\text{At } T = 0.9T_C, && N_0 = 0.146185N = 1461 && (9)\\ &\text{At } T = 0.1T_C, && N_0 = 0.968377N = 9683 && (10) \end{aligned}$$

■

Problem 2 [4 points]

Assume that I have a non-relativistic bosonic gas in $d = 2$ spatial dimensions. Use the fact that the density of states is constant in this case to show that Bose-Einstein condensation does not occur no matter how low the temperature gets.

Let's first show that the density of states is indeed constant. Using $p = \hbar k$, the energy of a k-state is

$$\begin{aligned} E &= \frac{p^2}{2m} \\ &= \frac{\hbar^2 k^2}{2m} \end{aligned} \tag{11}$$

Where in 2D we have $k^2 = k_x^2 + k_y^2$. Quantization leads to $k_i = 2\pi n_i/L$ (within the bounds of a 2D square box of side L). The k-space area for a given state is

$$\begin{aligned} \Delta k_x \Delta k_y &= \frac{2\pi}{L}\frac{2\pi}{L} \\ &= \frac{4\pi^2}{A} \end{aligned} \tag{12}$$

Where $A = L^2$ is total area of the square box. The (differential) number of states contained in the (differential) area d^2k in k-space is $d^2k/\Delta k_x \Delta k_y$. With this in mind, the total number of states within a domain in k-space is:

$$N_{\text{states}} = g_s \int \frac{A}{4\pi^2} d^2k \tag{13}$$

Where g_s is the number of spin degrees of freedom (degeneracy). We can now move from k-space to energy space. In polar (2D spherical) coordinates we have $d^2k = 2\pi k dk$ (angular part integrated out). Similarly, The $E \leftrightarrow k$ relation from above tells us that

$$k = \sqrt{\frac{2mE}{\hbar^2}} \quad \Rightarrow \quad k dk = \frac{m}{\hbar^2} dE \tag{14}$$

So

$$N = g_s \int \frac{Am}{2\pi\hbar^2} dE \tag{15}$$

From which we can read off the density of states (as a function of energy):

$$g(E) = g_s \frac{Am}{2\pi\hbar^2} \equiv g_0 \tag{16}$$

Which is independent of E, hence constant.

With this in mind, recall that the total number of particles (bosons) in our system is

$$N = \int_0^\infty \frac{g(E) dE}{e^{\beta(E-\mu)} - 1} \tag{17}$$

(It is important to distinguish the meaning between N and N_{states}. The first is the particles in the system, while the second counts the number of available states up to some energy level). If we let $g(E) = g_0$ be a constant, then the integral for N has yields

$$N = -\frac{g_0}{\beta}\ln(1-z) \tag{18}$$

Where $z = e^{\beta\mu}$ is the fugacity. Since $\mu < 0$, we see that this quantity is bounded:

$$z \in (0,1) \tag{19}$$

Fix N. Start cooling the system, so that T goes down. This makes β increase. At this rate, we need $\ln(1-z)$ to increase as well to keep N constant. What does this say about z? Solving for z, we see that

$$z = 1 - e^{-N\beta/g_0} \equiv 1 - e^{-\text{const.}\beta} \tag{20}$$

With β growing (in fact, we may take the $\beta \to \infty$ limit), we see that

$$z \to 1 \tag{21}$$

But it will never reach $z = 1$ exactly, as we had for the non-relativistic gas in $d = 3$ spatial dimensions. The constant nature of g_0 in eq. 20 allows to to drop the temperature indefinitely without ever hitting $z = 1$, the condition for BEC. ■

Problem 3 [6 points]

Each atom in a chunk of copper contributes one conduction electron. Look up the density and atomic mass of copper, and calculate the Fermi energy, the Fermi temperature, and the degeneracy pressure. Is room temperature sufficiently low to treat this system as a degenerate electron gas?

The density of copper is

$$\rho_{\text{Cu}} = 8.96 \times 10^3 \text{ kg/m}^3 \ , \tag{22}$$

and its atomic mass is

$$M = 6.3546 \times 10^{-2} \text{ kg/mol} \tag{23}$$

Using Avogadro's constant[1], we can compute the number density of conduction electrons

$$\begin{aligned} n_{\text{Cu,e}} &= \frac{\rho N_A}{M} \\ &= 8.49123 \times 10^{28} \text{ m}^{-3} \end{aligned} \tag{24}$$

Then by direct computation, we have

1. Fermi energy.

   Recalling that $g_s = 2$, and using the mass of electrons and the number density of conduction electrons...

$$\begin{aligned} E_F &= \frac{h^2}{2m}\left(\frac{3}{8\pi}n\right)^{2/3} \\ &= 1.12979 \times 10^{-18} \text{ Joules} \\ &\equiv 7.05159 \text{ eV} \end{aligned} \tag{25}$$

2. Fermi temperature.[2]

$$\begin{aligned} T_F &= \frac{E_F}{k_B} \\ &= 81,833.5 \text{ Kelvin} \end{aligned} \tag{26}$$

3. Degeneracy pressure.

$$\begin{aligned} P &= \frac{2}{5}nE_F \\ &= 3.83734 \times 10^{10} \text{ J/m}^3 \\ &= 3.83734 \times 10^{10} \text{ Pa} \\ &= 378,716.012 \text{ Atm} \end{aligned} \tag{27}$$

At room temperature, which we can approximate by[3] $T \simeq 300$ kelvin, we see that the Fermi temperature is much higher. Hence, we can treat this system as a degenerate electron gas. ■

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[1] $N_A = 6.02214 \times 10^{23} \text{ mol}^{-1}$.

[2] It is convenient to use Boltzmann's constant in units of $k_B = 8.617 \times 10^{-5}$ eV/Kelvin.

[3] This amounts to 80.33°F.