PHYS301
Homework 11 solutions

Spring 2026

Due: May 4th

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Problem 1 [5 points]

Thermal pollution: A river with water temperature of $T = 20°$ C is to be used as the low temperature reservoir of a large power plant, with a steam temperature of $T = 500°$ C. If ecological considerations limit the amount of heat that can be dumped into the river to 1500 Megawatts, what is the largest electrical output that the plant can deliver? If improvements in hot-steam technology would permit raising T by $100°$ C, what effect would this have on the plant capacity?

Let's model this system using a carnot cycle. Given ecological considerations, we compute the maximum amount of energy we can input into the plant as:

$$Q_{\mathrm{H}} = \frac{Q_{\mathrm{C}}}{1-\eta} \tag{1}$$

Where the efficiency of the cycle, η, can be computed using the provided temperatures (in Kelvin)

$$\begin{aligned} \eta &= 1 - \frac{T_{\mathrm{C}}}{T_{\mathrm{H}}} \\ &\approx 0.62 \end{aligned} \tag{2}$$

Hence

$$Q_{\mathrm{H}} \approx 3956.08 \text{ MW} \tag{3}$$

So the electrical output from the plant is

$$\begin{aligned} W &= \eta \cdot Q_{\mathrm{H}} \\ &= 2456.08 \text{ MW} \end{aligned} \tag{4}$$

If we instead work the cycle at a higher temperature of $T_{\mathrm{H}} = 500° + 100°$ C $= 873.15$ Kelvin, the efficiency improves slightly $\eta = 0.6642$, which allows us to burn more $Q_{\mathrm{H}} = 4467.77$ MW, yielding a higher (an increase of +20%) electric output 2967.77 MW. ■

Problem 2 [5 points]

A room air conditioner operates as a Carnot cycle refrigerator between an outside temperature T_h, and a room at a lower temperature T_l. The room gains heat from the outdoors at a rate $A(T_h - T_l)$; this heat is removed by the air conditioner. The power supplied to the cooling unit is P.

a) Show that the steady state temperature of the room is

$$T_l = \left(T_h + \frac{P}{2A}\right) - \sqrt{\left(T_h + \frac{P}{2A}\right)^2 - T_h} \tag{5}$$

b) If the outdoors is at 37° C and the room is maintained at 17° C by a cooling power of 2 kilowatts, find the heat loss coefficient A of the room in Watts/Kelvin.

a) The Carnot cycle satisfies

$$\frac{Q_l}{Q_h} = \frac{T_l}{T_h} \tag{6}$$

In the steady-state limit, with fixed T_l, T_h, we may write the above as

$$\frac{\dot{Q}_l}{\dot{Q}_h} = \frac{T_l}{T_h} \tag{7}$$

where the rate at which the room gains heat from the outdoors, $\dot{Q}_l \equiv dQ_l/dt$, is $A(T_h - T_l)$. Given power P (energy per unit time), the closed flow of the system follows

$$\dot{Q}_h = \dot{Q}_l + P \tag{8}$$

We can use this to remove $\dot{Q}_h$ dependence and observe that

$$\dot{Q}_l = \frac{PT_l}{T_h - T_l} \tag{9}$$

Using the provided rate, it follows that

$$A(T_h - T_l) = P\frac{T_l}{T_h - T_l} \tag{10}$$

This is a quadratic in T_l:

$$\begin{aligned} A(T_h - T_l)^2 &= PT_l \\ A(T_h^2 - 2T_hT_l + T_l^2) &= PT_l \\ (A)T_l^2 \ - \ (2AT_h + P)T_l \ + \ (AT_h^2) &= 0 \end{aligned} \tag{11}$$

Using the (−) solution to the general formula for quadratics, we see that

$$T_l = \left(T_h + \frac{P}{2A}\right) - \sqrt{\left(T_h + \frac{P}{2A}\right)^2 - T_h} \tag{12}$$

b) From part (a), we can solve $A(T_h - T_l) = P\frac{T_l}{T_h - T_l}$ for A:

$$A = P\frac{T_l}{(T_h - T_l)^2} \tag{13}$$

Using the provided values ($P = 2000$ Watts, $T_l = 290.15$ Kelvin, $T_h = 310.15$ Kelvin), we see that

$$A \simeq 1450\,\frac{\text{Watts}}{\text{Kelvin}} \tag{14}$$

■

Problem 3 [5 points]

Geothermal energy: A very large mass M of porous hot rock is to be utilized to generate electricity by injecting water and utilizing the resulting hot seam to drive a turbine. As a result of heat extraction, the temperature of the rock drops, according to

$$đQ_h = -MCdT_h \tag{15}$$

where C is the specific heat of the rock, assumed to be temperature independent.

a) If the plant operates at the Carnot limit, calculate the total amount W of electrical energy extractable from the rock, if the temperature of the rock was initially $T_h = T_i$, and if the plant is to be shut down when the temperature has dropped to $T_\mathrm{h} = T_\mathrm{f}$. Assume that the lower reservoir temperature T_l, stays constant.

b) Give a numerical value, in kilowatt-hours (kWh), for $M = 10^{14}$ kg (about 30 km^3), $C = 1$ J/g/K, $T_i = 600°$ C, $T_f = 110°$ C, and $T_l = 20°$ C. Be careful with units! Compare this number to the total electricity produced in the world in 2024, which was about 30850 TWh (1 TWh = 10^{12} Wh).

a) Given the efficiency of the cycle, η, the infinitesimal work dW produced due to an infinitesimal amount of heat extracted $đQ_\mathrm{h}$ is

$$dW = \eta đQ_\mathrm{h} \tag{16}$$

Since $\eta = 1 - (T_l/T_h)$ and using the provided $đQ_\mathrm{h}$, we integrate both sides to find the work produced (energy to be extracted) by the system is

$$\begin{aligned} W &= \int_{T_i}^{T_f} \left(1 - \frac{T_l}{T_h}\right)(-MC)dT_h \\ &= MC \int_{T_f}^{T_i} \left(1 - \frac{T_l}{T_h}\right) dT_h \\ &= MC \left[T_i - T_f - T_l \ln\left(\frac{T_i}{T_f}\right)\right] \end{aligned} \tag{17}$$

b) Converting kilograms to grams and celcius to Kelvin:

$$M = 10^{17} \text{ g} \qquad T_i = 873.15 \text{ Kelvin}$$
$$T_l = 293.15 \text{ Kelvin} \qquad T_f = 383.15 \text{ Kelvin}$$

Using the result from part (a), it follows that

$$\begin{aligned} W &= 2.48 \times 10^{19} \text{ Joules} \\ &= 6.9 \times 10^{12} \text{ kWh} \end{aligned} \tag{18}$$

In 2024 the total electricity produced was 30850 TWh $\equiv 30.85 \times 10^{12}$ kWh. Therefore, this system produces roughly $\sim 20\%$ of that. ■