Homework #12 Solutions

PHYS 301

Problem 1: Thermodynamics of Self-Interacting Dark Matter (SIDM)

(a) The Partition Function

The canonical partition function for a gas of N particles is:

$$Z(N,V,T) = \frac{1}{N!\lambda_Q^{3N}} \int d^{3N}r \prod_{i<j} e^{-\beta U(r_{ij})} \tag{1}$$

We introduce the Mayer f-function, $f(r) = e^{-\beta U(r)} - 1$. For a hard-sphere potential, $U(r) = \infty$ for $r < r_0$ and $U(r) = 0$ for $r \geq r_0$. Thus:

$$f(r) = \begin{cases} -1 & r < r_0 \\ 0 & r \geq r_0 \end{cases} \tag{2}$$

In the dilute limit, we approximate the product over pairs by retaining only the first-order term:

$$\prod_{i<j}(1 + f(r_{ij})) \approx 1 + \sum_{i<j} f(r_{ij}) \tag{3}$$

Integrating over the spatial coordinates:

$$\begin{aligned} \int d^{3N}r \left(1 + \sum_{i<j} f(r_{ij})\right) &= V^N + \frac{N(N-1)}{2} V^{N-1} \int d^3r f(r) \\ &\approx V^N \left(1 + \frac{N^2}{2V} \int d^3r f(r)\right) \end{aligned} \tag{4}$$

The integral of the f-function is simply the negative of the hard-sphere volume, which we define as $v_0 = \frac{4}{3}\pi r_0^3$:

$$\int_0^{r_0} 4\pi r^2(-1)dr = -\frac{4}{3}\pi r_0^3 = -v_0 \tag{5}$$

Using the binomial approximation $(1+x)^N \approx 1 + Nx$, we rewrite the configuration integral and obtain the partition function:

$$Z(N,V,T) \approx \frac{V^N}{N!\lambda_Q^{3N}} \left(1 - \frac{Nv_0}{2V}\right)^N = Z_{\text{ideal}} \left(1 - \frac{Nv_0}{2V}\right)^N \tag{6}$$

(b) Equation of State

The Helmholtz free energy is $F = -k_B T \ln Z$:

$$\begin{aligned} F &= -k_B T \ln \left[Z_{\text{ideal}} \left(1 - \frac{N v_0}{2V}\right)^N \right] \\ &= F_{\text{ideal}} - N k_B T \ln \left(1 - \frac{N v_0}{2V}\right) \end{aligned} \tag{7}$$

Using the Taylor expansion $\ln(1-x) \approx -x$ for small x:

$$F \approx F_{\text{ideal}} + \frac{N^2 k_B T v_0}{2V} \tag{8}$$

The pressure is $P = -\left(\frac{\partial F}{\partial V}\right)_{T,N}$:

$$\begin{aligned} P &= -\frac{\partial F_{\text{ideal}}}{\partial V} - \frac{\partial}{\partial V}\left(\frac{N^2 k_B T v_0}{2V}\right) \\ &= \frac{N k_B T}{V} + \frac{N^2 k_B T v_0}{2V^2} \end{aligned} \tag{9}$$

$$\boxed{P = \frac{N k_B T}{V}\left(1 + \frac{N v_0}{2V}\right)} \tag{10}$$

Since N, V, v_0, and T are all positive quantities, the pressure of the SIDM halo is strictly greater than the ideal CDM gas ($P > P_{\text{ideal}}$).

(c) Entropy

Entropy is $S = -\left(\frac{\partial F}{\partial T}\right)_{V,N}$:

$$\begin{aligned} S &= -\frac{\partial}{\partial T}\left(F_{\text{ideal}} + \frac{N^2 k_B T v_0}{2V}\right) \\ &= S_{\text{ideal}} - \frac{N^2 k_B v_0}{2V} \end{aligned} \tag{11}$$

$$\boxed{S(T,V) = S_{\text{ideal}} - \frac{N^2 k_B v_0}{2V}} \tag{12}$$

(d) Adiabatic Collapse

For an adiabatic process, $\Delta S = 0$, so $dS = 0$. Recall that $S_{\text{ideal}} = N k_B \ln(V T^{3/2}) + \text{const}$. Taking the differential of our entropy expression:

$$dS = d(S_{\text{ideal}}) - d\left(\frac{N^2 k_B v_0}{2V}\right) = 0 \tag{13}$$

$$N k_B \left(\frac{dV}{V} + \frac{3}{2}\frac{dT}{T}\right) + \frac{N^2 k_B v_0}{2V^2} dV = 0 \tag{14}$$

Divide out Nk_B:

$$\frac{3}{2}\frac{dT}{T} + \left(\frac{1}{V} + \frac{Nv_0}{2V^2}\right) dV = 0 \tag{15}$$

Rearranging to solve for the temperature-volume relationship:

$$\frac{dT}{T} = -\frac{2}{3}\frac{dV}{V}\left(1 + \frac{Nv_0}{2V}\right) \tag{16}$$

For a collapse, $dV < 0$, which makes dT positive (the halo heats up). Because the correction factor $(1 + \frac{Nv_0}{2V})$ is strictly greater than 1, $|dT|$ is larger for the SIDM halo than for the CDM halo.

Physical Reason: The hard-core repulsion effectively reduces the available "free volume" for the particles. Compressing this gas requires more work than an ideal gas because the particles are physically bumping into each other's excluded volumes, converting more mechanical work into thermal energy.

Problem 2: Irreversible Expansion of an Ultra-Relativistic Fermi Gas

(a) Quantum Microstate

For ultra-relativistic particles, $E = pc$. The number of states with momentum up to p is $\Phi(p) = \frac{g_s V}{h^3}\frac{4}{3}\pi p^3$. Using $p = E/c$ and $g_s = 2$:

$$\Phi(E) = \frac{8\pi V}{3h^3c^3}E^3 \tag{17}$$

The density of states $g(E) = \frac{d\Phi}{dE}$ is:

$$g(E) = \frac{8\pi V}{h^3c^3}E^2 \tag{18}$$

At $T = 0$, all states up to the Fermi energy $E_{F,i}$ are filled. The total number of particles N is:

$$N = \int_0^{E_{F,i}} g(E)dE = \frac{8\pi V_i}{3h^3c^3}E_{F,i}^3 \implies \boxed{E_{F,i} = hc\left(\frac{3N}{8\pi V_i}\right)^{1/3}} \tag{19}$$

The total internal energy is:

$$\begin{aligned} E_i &= \int_0^{E_{F,i}} Eg(E)dE = \frac{8\pi V_i}{h^3c^3}\int_0^{E_{F,i}} E^3 dE \\ &= \frac{8\pi V_i}{h^3c^3}\frac{E_{F,i}^4}{4} = \frac{2\pi V_i}{h^3c^3}E_{F,i}^4 \end{aligned} \tag{20}$$

To express this in terms of N, substitute $V_i/(h^3c^3) = 3N/(8\pi E_{F,i}^3)$:

$$E_i = 2\pi\left(\frac{3N}{8\pi E_{F,i}^3}\right)E_{F,i}^4 = \frac{3}{4}NE_{F,i} \tag{21}$$

(b) Degeneracy Pressure

Pressure is $P = -\left(\frac{\partial E_i}{\partial V}\right)_N$. We know $E_i = \frac{3}{4}NE_{F,i}$. From part (a), $E_{F,i} \propto V^{-1/3}$. Therefore, the derivative is:

$$\frac{\partial E_{F,i}}{\partial V} = -\frac{1}{3}\frac{E_{F,i}}{V} \tag{22}$$

Substitute this into the pressure equation:

$$P = -\frac{3}{4}N\left(-\frac{1}{3}\frac{E_{F,i}}{V}\right) = \frac{1}{4}\frac{NE_{F,i}}{V} \tag{23}$$

Since $E_i = \frac{3}{4}NE_{F,i}$, we can rewrite this as:

$$P = \frac{1}{3}\frac{E_i}{V} \implies PV = \frac{E_i}{3} \tag{24}$$

Since $E_i \propto V^{-1/3}$, it follows that $P \propto V^{-1}V^{-1/3} = V^{-4/3}$. Therefore:

$$\boxed{PV^{4/3} = \text{constant}} \tag{25}$$

(c) Free Expansion (First Law)

In a free expansion, $W = 0$ and $Q = 0$. By the First Law of Thermodynamics, $\Delta E = 0$, so the internal energy is conserved: $E_f = E_i$. In the final state, the gas is a classical ultra-relativistic ideal gas. By the equipartition theorem, the energy of a 3D gas with $E = pc$ is:

$$E_f = 3Nk_BT_f \tag{26}$$

Equating initial and final energies:

$$3Nk_BT_f = \frac{3}{4}NE_{F,i} \implies \boxed{T_f = \frac{E_{F,i}}{4k_B}} \tag{27}$$

(d) The Arrow of Time (Second Law)

To prove the process is strictly irreversible, we must show that the Clausius inequality for a cycle containing this process is strictly less than zero.

First, let us find the actual entropy difference between the final state f and initial state i. Even though the process is violent and non-equilibrium, entropy is a state function, so we can calculate $\Delta S = S_f - S_i$ by imagining a reversible path between them.

Using the fundamental thermodynamic identity:

$$TdS = dE + PdV \implies dS = \frac{1}{T}dE + \frac{P}{T}dV \tag{28}$$

From part (c), we know that the free expansion yields no net change in internal energy ($dE = 0$). Therefore, the entropy difference between the initial and final equilibrium states is:

$$S_f - S_i = \int_{V_i}^{V_f} \frac{P(V,T)}{T}dV \tag{29}$$

Because pressure P and absolute temperature T are strictly positive macroscopic quantities, and the gas expands into a vacuum ($V_f > V_i$, meaning $dV > 0$), the integral must evaluate to a strictly positive number. Therefore, $\mathbf{S_f} > \mathbf{S_i}$.

Now, consider a closed thermodynamic cycle consisting of Path I (the actual free expansion) and Path II (a reversible compression back to the initial state). The Clausius integral for this complete cycle is:

$$\oint \frac{\delta Q}{T} = \int_{\text{Path I}} \frac{\delta Q}{T} + \int_{\text{Path II}} \frac{\delta Q_{\text{rev}}}{T} \tag{30}$$

For the free expansion (Path I), no heat is exchanged ($\delta Q = 0$), so the first integral is exactly 0. For the reversible return (Path II), the integral is by definition the change in the system's entropy, which is $S_i - S_f$.

$$\oint \frac{\delta Q}{T} = 0 + (S_i - S_f) \tag{31}$$

Because we just proved that $S_f > S_i$, it must be true that $S_i - S_f < 0$. Therefore:

$$\oint \frac{\delta Q}{T} < 0 \tag{32}$$

The fact that the Clausius integral evaluates to strictly less than zero is the mathematical proof that the cycle contains irreversibility. Since Path II was explicitly defined as reversible, the free expansion (Path I) must be the source of the strict irreversibility.

Finally, the net change in the entropy of the universe is the sum of the entropy changes of the system and the environment. Since the environment exchanged no heat, $\Delta S_{\text{env}} = 0$. Assuming the fully degenerate initial state has $S_i = 0$ (per the Third Law of Thermodynamics), the net change in the entropy of the universe is simply the final entropy of the gas:

$$\Delta S_{\text{univ}} = \Delta S_{\text{system}} = S_f = \int_{V_i}^{V_f} \frac{P(V,T)}{T} dV > 0 \tag{33}$$