PHYS 301
Thermodynamics and Statistical Mechanics

Homework Assignment 1

Due date: Sunday January 31 2026 5pm, submitted online on UNM Canvas.

Question 1 (5 points).

Prove Sterling's formula. Start by noting that factorials are related to the Gamma function via $N! = \Gamma(N+1)$, with

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx. \tag{1}$$

Use the above to write

$$N! = \int_0^\infty e^{-F(x)} dx. \tag{2}$$

After establishing what the function $F(x)$ is, find its minimum (let's call it x_0). Then, Taylor expand $F(x)$ around its minimum

$$F(x) \approx F(x_0) + \frac{1}{2} F''(x_0)(x - x_0)^2, \tag{3}$$

and use this in the above integral to establish that

$$N! \approx \sqrt{2\pi N} N^N e^{-N}, \tag{4}$$

when $N \gg 1$. How accurate is Stirling's formula for $N = 10$? $N = 100$?

Question 2 (5 points).

Let's consider a system of N non-interacting spins, which each can be in either the spin up or spin down state. Let $N_\uparrow$ be the number of spins in the up state, and $N_\downarrow = N - N_\uparrow$ the number spins in the down state. Define the *spin excess* s as

$$2s \equiv N_\uparrow - N_\downarrow, \tag{5}$$

where the leading factor of 2 is just a convention.

(a) Starting from the multiplicity for a macrostate with $N_\uparrow$ spins up,

$$\Omega(N, N_\uparrow) = \frac{N!}{N_\uparrow!(N - N_\uparrow)!}, \tag{6}$$

show that the multiplicity of a macrostate with spin excess s is

$$\Omega(N, s) = \frac{N!}{(\frac{1}{2}N + s)!(\frac{1}{2}N - s)!}. \tag{7}$$

(b) In the limit that $s/N \ll 1$ and $N \gg 1$, show that this multiplicity is approximately Gaussian (up to a normalization factor)

$$\Omega(N, s) \simeq (2/(\pi N))^{1/2} 2^N e^{-2s^2/N}. \tag{8}$$

(c) What is the standard deviation (width) of this Gaussian? Use this information to show that the width to height ratio of the above multiplicity scales as

$$\sim \frac{N}{2^N} \tag{9}$$

for $N \gg 1$. Use this information to argue that the above multiplicity is *extremely* sharply peaked $s = 0$. If you were to draw in your homework the above $\Omega(N, s)$ function for $N = 1000$ with a height at $s = 0$ of 10cm, what would be the width of the multiplicity that you would draw?

Question 3 (4 points).

The meaning of "never." It has been said that "six monkeys, set to strum unintelligently on typewriters for millions of years, would be bound in time to write all the books in the British Museum." This statement is nonsense, for it gives a misleading conclusion about very, very large numbers. Could all the monkeys in the world have typed out a single specified book in the age of the universe?

Suppose that 10^{10} monkeys have been seated at typewriters throughout the age of the universe, 10^{18} s. This number of monkeys is about three times greater than the present human population of the earth. We suppose that a monkey can hit 10 typewriter keys per second. A typewriter may have 44 keys; we accept lowercase letters in place of capital letters. Assuming that Shakespeare's *Hamlet* has 10^5 characters, will the monkeys hit upon *Hamlet*?

(a) Show that the probability that any given sequence of 10^5 characters typed at random will come out in the correct sequence (the sequence of *Hamlet*) is of the order of

$$\left(\frac{1}{44}\right)^{100\,000} = 10^{-164\,345},$$

where we have used $\log_{10} 44 = 1.64345$.

(b) Show that the probability that a monkey-Hamlet will be typed in the age of the universe is approximately $10^{-164\,316}$. The probability of *Hamlet* is therefore zero in any operational sense of an event, so that the original statement at the beginning of this problem is nonsense: one book, much less a library, will never occur in the total literary production of the monkeys.