PHYS 301
Thermodynamics and Statistical Mechanics

Homework Assignment 12

Due date: Monday May 11 2026 5pm, submitted on UNM Canvas

Question 1 (10 points).

Thermodynamics of Self-Interacting Dark Matter (SIDM): Astrophysicists often model dark matter halos as a macroscopic gas of particles. In the standard Cold Dark Matter (CDM) model, these particles are entirely collisionless and can be treated as an ideal gas. However, in Self-Interacting Dark Matter (SIDM) models, particles have a finite scattering cross-section.

Let us model a spherical SIDM halo as a classical gas of N particles of mass m confined to a volume V at temperature T. To account for the interactions, assume the particles interact via a simple hard-sphere repulsive potential of radius r_0:

$$U(r) = \begin{cases} \infty & \text{for } r < r_0 \\ 0 & \text{for } r \geq r_0 \end{cases} \tag{1}$$

(a) **The Partition Function:** Assuming the gas is dilute ($V/N \gg r_0^3$), use the Mayer f-function to evaluate the configuration integral. Derive the leading-order correction to the canonical partition function $Z(N, V, T)$.

(b) **Equation of State:** Compute the Helmholtz free energy F for this halo. From this, derive the equation of state. Show that the pressure is strictly greater than that of a CDM (ideal) halo.

(c) **Entropy:** Calculate the entropy $S(T, V)$ of the SIDM gas. Express your answer in terms of the ideal gas entropy S_{ideal} and a correction term dependent on r_0.

(d) **Adiabatic Collapse:** Suppose the dark matter halo undergoes a slow, quasi-static adiabatic compression due to the infall of baryonic matter, shrinking from V_i to V_f. Using your entropy expression from part (c), determine the differential relationship between T and V. Does the SIDM halo heat up more rapidly or less rapidly than a collisionless CDM halo undergoing the exact same volume change? Explain the physical reason for this difference.

Question 2 (10 points).

Irreversible Expansion of an Ultra-Relativistic Fermi Gas: In the cores of certain stellar remnants, electrons are pushed to such extreme densities and momenta that they must be treated as a fully degenerate, ultra-relativistic Fermi gas where the energy-momentum relation is exactly $E = pc$.

(a) **Internal Energy:** Consider N non-interacting ultra-relativistic electrons (spin degeneracy $g_s = 2$) at $T = 0$ confined to a volume V_i. Calculate the Fermi energy $E_{F,i}$ as a function of the number density. Then, integrate over the density of states to show that the total internal energy is $E_i = \frac{3}{4}NE_{F,i}$.

(b) **Degeneracy Pressure:** Using the thermodynamic identity $P = -\left(\frac{\partial E}{\partial V}\right)_{N,S}$, derive the degeneracy pressure of this ultra-relativistic gas. Show that during a quasi-static volume change at zero temperature, the condition $PV^{4/3} = \text{constant}$ holds.

(c) **Free Expansion (First Law):** Suppose a cataclysmic astrophysical event causes the confining boundary of the remnant to vanish. The electron gas undergoes an irreversible free expansion into the vacuum of space, doing no work and exchanging no heat, eventually reaching a massive volume V_f. The expansion is large enough that quantum degeneracy is broken and the classical limit applies. Find the final temperature T_f of the gas. *(Hint: For a classical ultra-relativistic gas, the equipartition of energy yields $E = 3Nk_BT$.)*

(d) **The Arrow of Time (Second Law):** By evaluating the Clausius inequality ($\oint \frac{đQ}{T} \leq 0$), explain mathematically why this free expansion is strictly irreversible. Assuming the initial entropy of the fully degenerate state was exactly zero, write down the integral required to find the net change in the entropy of the universe (ΔS_{univ}) for this process. No need to evaluate the integral.