PHYS 301
Thermodynamics and Statistical Mechanics

Homework Assignment 5

Due date: Sunday March 8 2026 5pm, submitted on UNM Canvas

Question 1 (10 points).

You probably know that atmospheric pressure on Earth decreases as we go up in altitude. We can understand this behavior by modeling the atmosphere as different layers of monoatomic ideal gas that are in diffusive (chemical) and thermal equilibrium. Consider one such layer at a fixed height h above the surface of the Earth. It is made of an ideal gas at temperature T, with N atoms of mass m in volume V. Here, N will be a function of the height, that is, $N = N(h)$.

(a) By first computing the partition function Z, show that the Helmholtz free energy $F = -k_\mathrm{B}T \ln Z$ for this atmospheric layer is

$$F = -k_\mathrm{B}T \ln\left(\frac{V^N}{N!\lambda_Q^3}\right) + mghN, \tag{1}$$

where g is the gravitational acceleration, and λ_Q is the usual Thermal de Broglie wavelength.

(b) From the free energy, show that the chemical potential of an atmospheric layer at a fixed height h above the surface of the Earth is

$$\mu(h) = k_\mathrm{B}T \ln\left(n(h)\lambda_Q^3\right) + mgh, \tag{2}$$

where $n(h) \equiv N(h)/V$ is the number density of atoms at height h.

(c) Use the fact that the different atmospheric layers are in diffusive equilibrium $\mu(h_1) = \mu(h_2)$ to determine how $n(h)$ varies with altitude. Assume the temperature to be the same across all atmospheric layers (this is usually not true on Earth). It is easiest to choose $h_1 = 0$ and $h_2 = h$, and express your answer in terms of $n(0)$, the number density of gas at the Earth's surface.

(d) Use the ideal gas law to derive the pressure profile $P(h)$ of our atmosphere. Assuming that our atmosphere is made of nitrogen (N_2, $m = 48 \times 10^{-27}$ kg) at $T = 227$ K, plot $P(h)/P(0)$ as a function of h. What is $P(h)/P(0)$ at the top of mount Everest ($h \simeq 8,850$ m)? What about at the altitude at which modern planes fly ($h \simeq 11,300$ m)?

Question 2 (3 points).

A neutral gas consists of N_e electrons e^-, N_p protons $p+$ and N_H Hydrogen atoms H. An electron and proton can combine to form Hydrogen

$$e^- + p^+ \leftrightarrow H. \tag{3}$$

From the free energy of the system $F(T, V; N_e, N_p, N_H)$, we can define a chemical potential for each of the three species as

$$\mu_i = \frac{\partial F}{\partial N_i}, \tag{4}$$

where $i = e, p, H$. By minimizing the free energy at *fixed temperature and volume*, (i.e. set $dF = 0$ with $dT = dV = 0$), show that the condition for equilibrium is

$$\mu_e + \mu_p = \mu_H. \tag{5}$$

[*Hint: How are dN_e, dN_p, and dN_H related to each other? You don't need to derive an explicit expression for the free energy to do this problem.*]

Question 3 (4 points).

Consider a two-state system made of a ground state with energy 0 and an excited state with energy ϵ.

(a) Assuming that the system can be occupied by at most *one particle*, show that the grand canonical partition function (Gibbs sum) takes the form

$$\mathcal{Z} = 1 + \lambda + \lambda e^{-\beta\epsilon}, \tag{6}$$

where $\lambda = e^{\beta\mu}$, with μ being the chemical potential and $\beta = 1/(k_B T)$.

(b) Derive an expression for the average occupancy $\langle N \rangle$ of the system.

(c) Derive an expression for the average energy of the system.

(d) Now assume that the system can host at most *two particles*, show that the grand canonical partition function takes the form

$$\mathcal{Z} = 1 + \lambda + \lambda e^{-\beta\epsilon} + \lambda^2 e^{-\beta\epsilon} = (1 + \lambda)(1 + \lambda e^{-\beta\epsilon}). \tag{7}$$

What does the above factorization of the partition function tell you about the system?