

# PHYS 301

## Thermodynamics and Statistical Mechanics

### Homework Assignment 6

Due date: Sunday March 15 2026 5pm, submitted on UNM Canvas

#### Question 1 (10 points).

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This problem explores the connection between the canonical (standard) partition function  $Z$  and its cousin the grand partition function  $\mathcal{Z}$  (also called the Gibbs sum). Let  $Z_N$  be the canonical partition function for  $N$  particles.

- (a) Show that the **grand partition function**  $\mathcal{Z}$  can be written as

$$\mathcal{Z}(\mu, V, T) = \sum_{N=0}^{\infty} \lambda^N Z_N(V, T), \quad (1)$$

where  $\lambda = e^{\beta\mu}$  is referred to as the *fugacity*. Here,  $\mu$  is the chemical potential and  $\beta = 1/(k_B T)$ . We therefore see that the grand partition function can be written as a sum over the standard partition function of systems with different number of particles, each weighted by a different power of  $\lambda$ .

- (b) The fugacity allows us to write simple expressions for the average particle number  $\langle N \rangle$  and its variance  $(\Delta N)^2$ . Show that

$$\langle N \rangle = \lambda \frac{\partial}{\partial \lambda} \ln \mathcal{Z}, \quad (\Delta N)^2 = \langle N^2 \rangle - \langle N \rangle^2 = \left( \lambda \frac{\partial}{\partial \lambda} \right)^2 \ln \mathcal{Z}. \quad (2)$$

- (c) If  $Z_N = Z_1^N / N!$  (where  $Z_1$  is the partition function for one particle), show that

$$\mathcal{Z}(\mu, V, T) = e^{\lambda Z_1(V, T)}. \quad (3)$$

- (d) In this case, show that the particle number  $N$  in the system is very sharply peaked around  $\langle N \rangle$  when  $N$  is large, that is,

$$\frac{\Delta N}{\langle N \rangle} = \frac{1}{\langle N \rangle^{1/2}}. \quad (4)$$

#### Question 2 (4 points).

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Live cells from plants and animals are generally *not* in chemical equilibrium with their surrounding. For instance, the number density  $n = N/V$  of potassium  $K^+$  ions in the internal sap of a plant cell (for example, a fresh water alga) may exceed by a factor of  $10^4$  the number density of  $K^+$  ions in the pond water in which the cell is growing. The chemical potential of the  $K^+$  ions is thus much higher inside the cell than in its surrounding. Estimate the difference in chemical potential between

the cell's interior and the pond water surrounding it at  $T = 300$  K and show that it is equivalent to a voltage of 0.24 Volt across the cell wall. You can approximate the chemical potential  $\mu$  as that of an ideal gas.

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**Question 3** (8 points).

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Consider a system of five particles, inside a container where the allowed energy levels are nondegenerate and evenly spaced (that is, the energy levels are  $0, \epsilon, 2\epsilon, 3\epsilon, 4\epsilon, \text{etc.}$ ) In this problem you will consider the allowed states for this system, depending on whether the particles are identical fermions, identical bosons, or distinguishable particles.

- (a) Describe the ground state of this system, for each of these three cases.
- (b) Suppose that the system's total energy is  $\epsilon$ . Describe the allowed states of the system, for each of the three cases. How many possible system states are there in each case?
- (c) Repeat part (b) when the total energy of the system is  $2\epsilon$ .
- (d) Suppose that the temperature of this system is low, so that the total energy is low (though not necessarily zero). In what way will the behavior of the bosonic system differ from that of the system of distinguishable particles? Discuss.