PHYS 301
Thermodynamics and Statistical Mechanics

Homework Assignment 7

Due date: Monday March 30 2026 5pm, submitted on UNM Canvas

Question 1 (10 points).

In class, we saw that the partition function at temperature T for photons of a given (angular) frequency ω was given by

$$Z_\omega = \sum_{N=0}^{\infty} e^{-\beta N \hbar\omega} = \frac{1}{1 - e^{-\beta\hbar\omega}}. \tag{1}$$

Since photons don't interact with one another, the different photon frequency modes are independent of each other. The total partition function Z is thus simply the product of the partition function of different frequency modes ω_i

$$Z = \prod_i Z_{\omega_i} \tag{2}$$

(a) Use to above to compute the logarithm of the total partition function for a photon gas at temperature T in a box of volume V and show that is is equal to

$$\ln Z = -\frac{V}{\pi^2 c^3} \int_0^\infty d\omega\, \omega^2 \ln\left(1 - e^{-\hbar\omega/k_{\mathrm{B}}T}\right). \tag{3}$$

You will need the fact that the number of photon states in an infinitesimal phase-space volume $d^3x d^3p$ is

$$2\frac{d^3x d^3p}{h^3}, \tag{4}$$

together with the fact that the energy of a photon is $E = \hbar\omega = pc$, where p is the momentum.

(b) Using the Helmholtz free-energy $F = -k_{\mathrm{B}}T \ln Z$, compute the pressure and entropy of this photon gas. In particular, show that the pressure of the photon gas in the box is $p = \rho/3$, where ρ is the average energy density (energy per unit volume) in the box.

Question 2 (10 points).

The purpose of this question is to explain why the cosmic microwave background (CMB) radiation still has a black body spectrum, even though it has not been in thermal equilibrium with matter since very early in the universe's history. Consider a region of volume V in the cosmos containing black body radiation of temperature T. Suppose the cosmos expands (slowly) by a scale factor $a > 1$ (such that the volume increases such that $V \to a^3 V$), so that the momentum p and angular frequency ω of each electromagnetic radiation mode are rescaled by $1/a$.

(a) If there are no electrically charged particles around to emit or absorb photons, the total number of photons in the expanded volume is the same as the initial one. Use this fact to show that the temperature in the expanded volume is lower by a factor of $1/a$. Thus, the CMB cools as the Universe expands.

(b) Use this scaling of the temperature to show that the Planck function remains valid after the expansion (that is, it maintains its shape), up to an overall scaling factor. From this scaling factor, show that the energy density (energy per unit volume) is down by a factor of $1/a^4$ after the expansion.

(c) Using the expression for the entropy of a photon gas derived in question 1, show that the entropy in the expanded volume is the same as the original entropy, thus confirming that the expansion of the Universe does not create entropy (in other words, it is adiabatic).