Review of BEC and The Degenerate Fermi Gas

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1 Review of Bose-Einstein Condensation (BEC)

We previously considered a non-relativistic gas of bosons with N particles at temperature T. The total number of particles is fixed and is given by:

$$N = \frac{V}{\lambda_Q^3} g_{3/2}(z) \tag{1}$$

where:

- $\lambda_Q = \frac{h}{\sqrt{2\pi m k_B T}}$ is the thermal de Broglie wavelength.
- $z = e^{\beta\mu}$ is the fugacity.
- $g_{3/2}(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^{3/2}}$ is the Bose-Einstein integral.

Because the chemical potential must be negative ($\mu < 0$) for bosons, the fugacity is bounded: $z \in (0, 1)$.

As the temperature T decreases (so λ_Q increases), the function $g_{3/2}(z)$ must increase to keep the total particle number N fixed. But the largest possible value for $g_{3/2}(z)$ is bounded at $z = 1$, where $g_{3/2}(1) = \zeta(3/2) \simeq 2.612$.

There is a critical temperature, T_c, at which this occurs. Setting $z = 1$:

$$\frac{Nh^3}{V(2\pi m k_B T_c)^{3/2}} = \zeta(3/2) \simeq 2.61 \tag{2}$$

Solving for T_c:

$$\left[\frac{Nh^3}{V\zeta(3/2)}\right]^{2/3} \frac{1}{2\pi m k_B} = T_c \tag{3}$$

1.1 What about $T < T_c$?

Let us reconsider our original sum for the number of particles:

$$N = \sum_r \frac{1}{e^{\beta(E_r - \mu)} - 1} = \int_0^\infty \frac{g(E)\, dE}{e^{\beta(E-\mu)} - 1} \tag{4}$$

When we converted the sum to an integral, we used the density of states $g(E) \propto \sqrt{E}$. Because $g(0) = 0$, the integral assigns zero weight to the ground state ($E_0 = 0$).

For $T < T_c$, a macroscopic number of particles accumulates in the ground state. We must explicitly separate the ground state from the excited states:

$$N = N_0 + N_{ex} = \frac{1}{e^{-\beta\mu} - 1} + \int_0^\infty \frac{g(E)\, dE}{e^{\beta(E-\mu)} - 1} \tag{5}$$

For $T < T_c$, the chemical potential $\mu \to 0$, allowing N_0 to become macroscopic. The number of particles in the excited states reaches its maximum value:

$$N_{ex} = \frac{V}{\lambda_Q^3}\zeta(3/2) = N\left(\frac{T}{T_c}\right)^{3/2} \tag{6}$$

Thus, the macroscopic occupation of the ground state (the Bose-Einstein Condensate) is:

$$N_0 = N - N_{ex} = N\left[1 - \left(\frac{T}{T_c}\right)^{3/2}\right] \tag{7}$$

2 The Degenerate Fermi Gas

Now, what if we have a non-relativistic gas of fermions instead of bosons? For fermions, the average number of particles is governed by Fermi-Dirac statistics:

$$N = \sum_r \frac{1}{e^{\beta(E_r-\mu)} + 1} \tag{8}$$

Unlike bosons, there is no restriction on μ being negative. μ can be positive or negative.

Let us consider the absolute zero temperature limit ($T \to 0$, or $\beta \to \infty$). In this limit, the gas is highly degenerate. The Fermi-Dirac distribution $f(E) = \frac{1}{e^{\beta(E-\mu)}+1}$ becomes a sharp step function:

- $f(E) = 1$ for $E < \mu$
- $f(E) = 0$ for $E > \mu$

Because of the Pauli exclusion principle, each successive fermion we add to the system will occupy a higher energy state. The energy of the highest filled state at $T = 0$ is called the **Fermi energy**, denoted E_F:

$$E_F = \mu(T = 0) \tag{9}$$

Particles sitting exactly at the Fermi energy have the highest possible momentum, called the **Fermi momentum** (P_F):

$$P_F = \sqrt{2mE_F} \tag{10}$$

All occupied states have $p \leq P_F$; this collection of filled states forms the **Fermi sea**. The states exactly at $p = P_F$ form the **Fermi surface**.

2.1 Computing the Fermi Energy

We can compute the Fermi energy for a gas of N fermions. At $T = 0$, the number of particles is just the integral of the density of states up to the Fermi energy:

$$\begin{aligned} N &= \sum_r \frac{1}{e^{\beta(E_r-\mu)} + 1} \\ &= \frac{1}{h^3}\int d^3x \int d^3p \frac{1}{e^{\beta(E-\mu)} + 1} \\ &= \int_0^\infty \frac{g(E)\,dE}{e^{\beta(E-\mu)} + 1} \\ &= \int_0^{E_F} g(E)\,dE \end{aligned} \tag{11}$$

Recall the density of states $g(E)$:

$$g(E) = g_s \frac{4\pi\sqrt{2}m^{3/2}}{h^3} V\sqrt{E} \tag{12}$$

Here, g_s is the spin degeneracy factor to account for the internal states of the fermion (e.g., for electrons, spin-up and spin-down means $g_s = 2$).

Integrating the density of states from $E = 0$ to $E = E_F$:

$$\begin{aligned} N &= \int_0^{E_F} g_s \frac{4\pi\sqrt{2}m^{3/2}}{h^3} V E^{1/2}\, dE \\ &= g_s \frac{4\pi\sqrt{2}m^{3/2}}{h^3} V \left[\frac{2}{3}E^{3/2}\right]_0^{E_F} \\ &= g_s \frac{4\pi\sqrt{2}m^{3/2}}{h^3} V \left(\frac{2}{3}E_F^{3/2}\right) \end{aligned} \tag{13}$$

We can now rearrange this equation to solve for E_F:

$$E_F^{3/2} = \frac{3N}{2Vg_s} \frac{h^3}{4\pi\sqrt{2}m^{3/2}} \tag{14}$$

$$E_F = \left[\frac{3N}{2Vg_s} \frac{h^3}{4\pi\sqrt{2}m^{3/2}}\right]^{2/3} \tag{15}$$

Squaring the terms inside the bracket and simplifying:

$$\boxed{E_F = \frac{h^2}{2m}\left(\frac{3N}{4\pi V g_s}\right)^{2/3}} \tag{16}$$

This gives the maximum energy of a fermion in a fully degenerate gas purely as a function of the particle number density N/V and fundamental constants. We can define a related temperature, called the **Fermi Temperature**

$$\boxed{T_F = E_F/k_B}. \tag{17}$$

For the conduction electrons in a metal, this Fermi temperature can be quite large (e.g. $\sim 10^4$ K). In general, this rather large temperature (or Fermi energy) means that a degenerate fermion gas has significant pressure. We will derive this pressure in the worksheet.