

# Quantum Gases: Fermions vs. Bosons

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## 1 The Breakdown of the Classical Partition Function

We have seen that the partition function for a system of  $N$  indistinguishable, non-interacting particles is given by:

$$Z = \frac{1}{N!} Z_1^N \quad (1)$$

where  $Z_1$  is the single-particle partition function.

However, this expression is only correct if the particles are *always* in different states. To see why this breaks down, consider a simple system of two non-interacting particles that can occupy 5 available states, all of which have the same energy ( $E = 0$ ). Because all energy levels are zero, all Boltzmann factors are 1 ( $e^{-\beta(0)} = 1$ ), and the partition function is simply equal to the total number of accessible microstates:  $Z = \Omega$ .

- **If the particles are distinguishable:** Each particle has 5 possible states ( $Z_1 = 5$ ). The total partition function is:

$$Z_{dist} = 5 \times 5 = 25 \quad (2)$$

- **If the particles are indistinguishable:** The classical formula yields:

$$Z_{indist} = \frac{1}{2!} (Z_1)^2 = \frac{1}{2} (5^2) = 12.5 \quad (3)$$

This result ( $Z = 12.5$ ) cannot be right! The number of microstates must be an integer.

### 1.1 Counting the States Carefully

Let us list the states out explicitly to see where the formula fails. We have 5 states (represented by 5 slots) and 2 particles.

#### Case 1: Particles occupy DIFFERENT states

First, let's look at the configurations where the two particles are in different slots.

If the particles are **indistinguishable** (we just write a "1" in occupied slots):

11000 10100 10010 10001  
01100 01010 01001  
00110 00101  
00011

There are exactly  $\binom{5}{2} = 10$  such states.

If the particles are **distinguishable** (let's call them  $A$  and  $B$ ), every single one of those 10 states splits into 2 distinct microstates depending on which particle is in which slot (e.g.,  $AB000$  vs  $BA000$ ). Thus, there are  $10 \times 2 = 20$  states.

## Case 2: Particles occupy the SAME state

Now, look at the configurations where both particles pile into the exact same slot.

If the particles are **indistinguishable** (we write a "2" to denote two identical particles):

20000 02000 00200 00020 00002

There are exactly **5** such states.

If the particles are **distinguishable**, there are *still only 5 states*. Putting particle *A* and particle *B* together in the first slot (*AB0000*) is the exact same physical microstate as putting particle *B* and particle *A* in the first slot (*BA0000* is not a distinct state).

## The Total Count

Adding these up gives us the true partition functions:

$$Z_{dist} = 20 + 5 = \mathbf{25} \quad (4)$$

$$Z_{indist} = 10 + 5 = \mathbf{15} \quad (5)$$

Notice that our classical formula gave 12.5, which is exactly  $25/2!$ . The factor of  $\frac{1}{N!}$  only accurately corrects the partition function for the overcounting of states when the particles occupy *different* states ( $20/2! = 10$ ). It incorrectly divides the states where particles occupy the *same* state ( $5/2! = 2.5$ ).

Therefore, the classical  $\frac{1}{N!}$  correction is only valid if the probability of two particles occupying the same state is negligibly small.

## 2 When Do Quantum Effects Become Important?

This leads to two crucial questions:

1. How likely is it that two particles occupy the same state?
2. Are there quantum mechanical restrictions on particles occupying the same state?

### 2.1 Question 1: The Probability of Occupying the Same State

If the number of particles  $N$  in a system is much less than the number of available states ( $N \ll Z_1$ ), then it is highly unlikely that two particles will end up in the same state.

For an ideal gas, we know that  $Z_1 = \frac{V}{\lambda_Q^3}$ , where  $\lambda_Q$  is the thermal de Broglie wavelength. So, the condition  $N \ll Z_1$  translates to:

$$\lambda_Q^3 \ll \frac{V}{N} \quad (6)$$

This means the quantum volume associated with a particle ( $\lambda_Q^3$ ) must be much smaller than the specific volume available per particle ( $V/N$ ).

Let's test this for air at room temperature.

- The thermal volume is roughly  $\lambda_Q^3 \sim (0.02 \text{ nm})^3 = 8 \times 10^{-6} \text{ nm}^3$ .
- The available volume per particle is roughly  $V/N \sim (3 \text{ nm})^3 = 27 \text{ nm}^3$ .

Since  $8 \times 10^{-6} \ll 27$ , this condition is very well satisfied for standard room-temperature gases. The classical approximation holds.

However, for a **dense** or very **cold** gas, the de Broglie wavelength grows and the available volume shrinks. When:

$$\lambda_Q^3 \gtrsim \frac{V}{N} \quad (7)$$

there is a fair probability that multiple particles will attempt to occupy the same state. From a quantum perspective, this means the wave functions of the different particles are physically overlapping. Quantum effects become fundamentally important, and the system must be treated as a **quantum gas**.

## 2.2 Question 2: Restrictions on State Occupation

When quantum effects dominate, quantum mechanics imposes strict rules on whether two indistinguishable particles can occupy the exact same state. Particles in nature are divided into two fundamental classes based on their intrinsic spin:

- **Bosons:** Particles that *can* occupy the same state without restriction.
  - They possess integer spin ( $s = 0, 1, 2, \dots$ ).
  - *Examples:* Photons, Phonons, Helium-4 atoms.
- **Fermions:** Particles that *cannot* occupy the same state. They strictly obey the Pauli Exclusion Principle (no two identical fermions can occupy the exact same quantum state simultaneously).
  - They possess half-integer spin ( $s = 1/2, 3/2, 5/2, \dots$ ).
  - *Examples:* Electrons, Protons, Neutrons, Neutrinos, Helium-3 atoms.