Review: Bosons vs. Fermions and the Planck Distribution

PHYS 301: Prof. Cyr-Racine

1 Review: Bosons vs. Fermions

1.1 Bosons (Bose-Einstein Statistics)

Bosons are particles that can occupy the same quantum state without restriction. They have integer spin ($s = 0, 1, 2, \ldots$).

Consider a simple system at temperature T and chemical potential μ with a single accessible energy level, $E = \epsilon$. Since there is no restriction on occupancy, the number of particles in this state can be $N = 0, 1, 2, 3, \ldots$ (assuming $\mu < \epsilon$).

The grand canonical partition function $\mathcal{Z}$ is the sum over all possible number of particles:

$$\begin{aligned} \mathcal{Z} &= \sum_{N=0}^{\infty} e^{-\beta(N\epsilon - \mu N)} = \sum_{N=0}^{\infty} e^{-N\beta(\epsilon-\mu)} \\ &= \sum_{N=0}^{\infty} \left(e^{-\beta(\epsilon-\mu)}\right)^N \end{aligned}$$

This is a geometric series. Summing the series yields:

$$\mathcal{Z} = \frac{1}{1 - e^{-\beta(\epsilon-\mu)}} \tag{1}$$

The average number of particles $\langle N \rangle$ in this state is given by the derivative of $\ln \mathcal{Z}$:

$$\begin{aligned} \langle N \rangle_{\text{bosons}} &= \frac{1}{\beta}\frac{\partial \ln \mathcal{Z}}{\partial \mu} \\ &= \frac{1}{\beta}\frac{1}{\mathcal{Z}}\frac{\partial \mathcal{Z}}{\partial \mu} \\ &= \frac{1}{\beta}\left(1 - e^{-\beta(\epsilon-\mu)}\right)\left[-\left(1 - e^{-\beta(\epsilon-\mu)}\right)^{-2}\left(-e^{-\beta(\epsilon-\mu)} \cdot \beta\right)\right] \\ &= \frac{e^{-\beta(\epsilon-\mu)}}{1 - e^{-\beta(\epsilon-\mu)}} \end{aligned}$$

Dividing the numerator and denominator by $e^{-\beta(\epsilon-\mu)}$, we obtain the **Bose-Einstein Distribution**:

$$\boxed{f_{\text{BE}} \equiv \langle N \rangle_{\text{bosons}} = \frac{1}{e^{\beta(\epsilon-\mu)} - 1}} \tag{2}$$

Note: States with $\epsilon \approx \mu$ are preferably occupied. For $\epsilon \gg k_B T$, $f_{\text{BE}} \sim e^{-\beta\epsilon}$, and states with large energies are exponentially suppressed.

1.2 Fermions (Fermi-Dirac Statistics)

Fermions are particles that cannot occupy the same quantum state due to the Pauli Exclusion Principle. They possess half-integer spin ($s = 1/2, 3/2, \dots$).

Considering the same single-level system ($E = \epsilon$), a state can only be either unoccupied ($N = 0$) or occupied by one particle ($N = 1$). The grand partition function is simply a sum of two terms:

$$\mathcal{Z} = 1 + e^{-\beta(\epsilon-\mu)} \tag{3}$$

The average occupancy is:

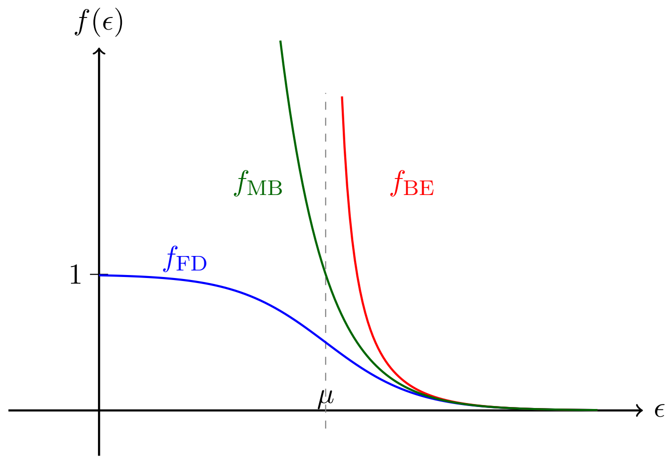
$$\begin{aligned}\langle N\rangle_{\text{fermions}} &= \frac{1}{\beta}\frac{\partial \ln \mathcal{Z}}{\partial \mu} \\ &= \frac{1}{\beta}\frac{1}{\mathcal{Z}}\left(e^{-\beta(\epsilon-\mu)} \cdot \beta\right) \\ &= \frac{e^{-\beta(\epsilon-\mu)}}{1 + e^{-\beta(\epsilon-\mu)}}\end{aligned}$$

Dividing by $e^{-\beta(\epsilon-\mu)}$ yields the **Fermi-Dirac Distribution**:

$$\boxed{f_{\text{FD}} \equiv \langle N\rangle_{\text{fermions}} = \frac{1}{e^{\beta(\epsilon-\mu)} + 1}} \tag{4}$$

Note: For states with $\epsilon < \mu$, the occupancy $f_{\text{FD}} \to 1$ (states are mostly occupied). For $\epsilon \gg \mu$, the occupancy is exponentially suppressed.

1.3 The Classical Limit



For both distributions, when the energy is much larger than the thermal energy ($\frac{\epsilon-\mu}{k_B T} \gg 1$), the exponential term dominates the denominator. In this limit:

$$f_{\text{BE}} \simeq f_{\text{FD}} \simeq e^{-\beta(\epsilon-\mu)} \equiv f_{\text{MB}} \tag{5}$$

This is the **classical limit** (the Maxwell-Boltzmann distribution), where the probability of occupancy is so low that quantum statistics don't matter ($Z_1 \gg N$).

2 Photons and the Planck Distribution

Consider a gas of photons. Photons are bosons (spin 1) and are massless. They are the fundamental quanta of the electromagnetic field.

- **Energy:** $E = \hbar\omega = pc$ (where p is momentum and c is the speed of light).
- **Chemical Potential:** Photons are not conserved. For example, a charged particle can emit or absorb a photon: $e^- \leftrightarrow e^- + \gamma$. In thermal equilibrium, the chemical potentials must balance: $\mu_e = \mu_e + \mu_\gamma$, which strictly implies:

$$\boxed{\mu_\gamma = 0} \tag{6}$$

Because $\mu = 0$, the occupancy of a photon state with energy $\hbar\omega$ at temperature T is given purely by the Bose-Einstein distribution:

$$\langle N \rangle_{\hbar\omega} = \frac{1}{e^{\hbar\omega/k_B T} - 1} \tag{7}$$

2.1 Average Energy and Phase Space Integration

In a macroscopic box at temperature T, many frequency modes ω will be occupied. To find the total average energy, we must sum over all possible states by integrating over phase space:

$$\langle E \rangle_{\text{box}} = \int \langle E \rangle_{\hbar\omega}\, g_s \frac{d^3x\, d^3p}{h^3} \tag{8}$$

where $g_s = 2$ accounts for the two independent polarization states of a photon.

Evaluating the spatial integral trivially yields the volume of the box ($\int d^3x = V$). We can transition the momentum integral into spherical coordinates ($d^3p = 4\pi p^2 dp$) and substitute $p = \hbar\omega/c$:

$$\begin{aligned} \langle E \rangle_{\text{box}} &= \frac{2V}{h^3} \int_0^\infty d^3p \frac{pc}{e^{pc/k_B T} - 1} \\ &= \frac{2V}{h^3} \int_0^\infty (4\pi p^2 dp) \frac{pc}{e^{pc/k_B T} - 1} \end{aligned}$$

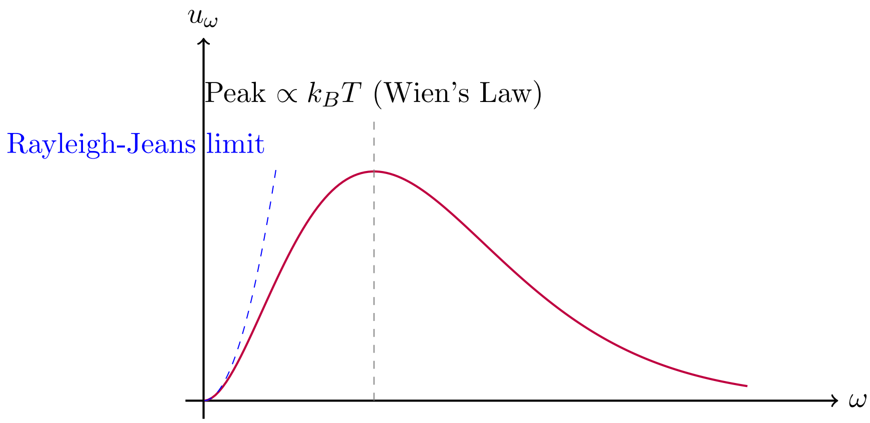
Substituting $p = \hbar\omega/c$ and $dp = (\hbar/c)d\omega$, and noting that $h^3 = (2\pi\hbar)^3 = 8\pi^3\hbar^3$:

$$\begin{aligned} \langle E \rangle_{\text{box}} &= \frac{8\pi V}{8\pi^3\hbar^3} \int_0^\infty \left(\frac{\hbar\omega}{c}\right)^2 \left(\frac{\hbar}{c} d\omega\right) \frac{\hbar\omega}{e^{\hbar\omega/k_B T} - 1} \\ &= \frac{V\hbar}{\pi^2 c^3} \int_0^\infty \frac{\omega^3}{e^{\hbar\omega/k_B T} - 1} d\omega \end{aligned}$$

2.2 The Spectral Energy Density

The integrand in the previous equation represents the energy per unit volume per unit frequency. This is the **Spectral Energy Density** (or Planck function):

$$\boxed{u_\omega = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{e^{\hbar\omega/k_B T} - 1}} \tag{9}$$



2.3 Total Energy Density and Stefan-Boltzmann Law

To find the total energy density $\rho_\gamma = \langle E \rangle / V$, we compute the full integral. We use the dimensionless variable $x = \hbar\omega/k_B T$, meaning $d\omega = (k_B T/\hbar)dx$:

$$\begin{aligned}\rho_\gamma &= \frac{\hbar}{\pi^2 c^3} \int_0^\infty \frac{(x k_B T/\hbar)^3}{e^x - 1} \left(\frac{k_B T}{\hbar} dx \right) \\ &= \frac{(k_B T)^4}{\pi^2 c^3 \hbar^3} \int_0^\infty \frac{x^3}{e^x - 1} dx\end{aligned}$$

The definite integral is a standard result equal to $\pi^4/15$. Thus, the total energy density of the photon gas is:

$$\boxed{\rho_\gamma = \frac{\pi^2 (k_B T)^4}{15(\hbar c)^3}} \tag{10}$$

This scales precisely as T^4.

If we want to find the energy flux J (energy per unit area per unit time) emitted from a hole in the box, we integrate the energy density over a hemisphere, accounting for the angle of incidence:

$$J = \int_{\text{hemisphere}} \left(\frac{\rho_\gamma}{4\pi} \right) c \cos\theta \, d\Omega = \frac{\rho_\gamma c}{4\pi} \int_0^{2\pi} d\phi \int_0^{\pi/2} \sin\theta \cos\theta \, d\theta$$

Evaluating the integrals yields:

$$J = \frac{\rho_\gamma c}{4\pi} (2\pi) \left[\frac{\sin^2\theta}{2} \right]_0^{\pi/2} = \frac{\rho_\gamma c}{4} = \sigma T^4 \tag{11}$$

This is the famous Stefan-Boltzmann Law.