Blackbody Spectrum and the Stefan-Boltzmann Law

Class Notes

1 The Blackbody Spectrum

Last time, we derived the Planck function (or Planck distribution), which describes the spectral energy density of a gas of photons. The energy per unit volume per unit angular frequency is given by:

$$u_\omega = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{e^{\hbar\omega/k_B T} - 1} \tag{1}$$

The total energy density ρ_γ is the integral of this distribution over all possible frequencies:

$$\rho_\gamma = \frac{\langle E \rangle}{V} = \int_0^\infty u_\omega \, d\omega = \frac{\pi^2 (k_B T)^4}{15(\hbar c)^3} \tag{2}$$

1.1 Wavelength Representation

Often, the spectrum is written in terms of wavelength λ rather than angular frequency ω. We can transform the distribution using the fundamental relation:

$$\lambda = \frac{2\pi c}{\omega} \implies d\lambda = -\frac{2\pi c}{\omega^2} \, d\omega \tag{3}$$

To conserve the total energy in a given interval, we require $u_\omega |d\omega| = u_\lambda |d\lambda|$. Using $|d\omega| = \frac{2\pi c}{\lambda^2} d\lambda$ and $\omega = \frac{2\pi c}{\lambda}$, we can substitute these into the Planck function. Note also that the photon energy is $\hbar\omega = \hbar\frac{2\pi c}{\lambda} = \frac{hc}{\lambda}$.

$$\begin{aligned} u_\lambda d\lambda &= u_\omega \left(\frac{2\pi c}{\lambda^2}\right) d\lambda \\ &= \frac{\hbar}{\pi^2 c^3} \left(\frac{2\pi c}{\lambda}\right)^3 \frac{1}{e^{hc/\lambda k_B T} - 1} \left(\frac{2\pi c}{\lambda^2}\right) d\lambda \end{aligned}$$

Recalling that $\hbar = h/2\pi$, we can simplify the prefactor:

$$\frac{h/2\pi}{\pi^2 c^3} \cdot \frac{8\pi^3 c^3}{\lambda^3} \cdot \frac{2\pi c}{\lambda^2} = \frac{8\pi hc}{\lambda^5}$$

This gives the spectral energy density per unit wavelength:

$$\boxed{u_\lambda = \frac{8\pi hc}{\lambda^5} \frac{1}{e^{hc/\lambda k_B T} - 1}} \tag{4}$$

The total energy density can equivalently be found by integrating over wavelength: $\rho_\gamma = \int_0^\infty u_\lambda d\lambda$.

Note: Some references discuss the Planck function in terms of specific intensity or radiance, denoted by B. The relations are:

$$B_\lambda = \frac{c}{4\pi} u_\lambda \quad \text{and} \quad B_\omega = \frac{c}{4\pi} u_\omega \tag{5}$$

2 Energy Flux and the Stefan-Boltzmann Law

What is the energy flux J (energy per unit area per unit time) escaping through a small hole in the cavity containing our photon gas?

To find this, we must integrate the energy density escaping over a hemisphere, taking into account the angle of incidence θ (which introduces a $\cos\theta$ geometrical factor) and the speed of light c:

$$J = \int_{\text{hemisphere}} \frac{\rho_\gamma}{4\pi} c \cos\theta \, d\Omega \tag{6}$$

Using the solid angle element $d\Omega = \sin\theta \, d\theta \, d\phi$, the integral over the hemisphere (θ from 0 to $\pi/2$, ϕ from 0 to 2π) becomes:

$$J = \frac{\rho_\gamma c}{4\pi} \int_0^{2\pi} d\phi \int_0^{\pi/2} \cos\theta \sin\theta \, d\theta \tag{7}$$

The azimuthal integral yields 2π. For the polar integral, let $u = \sin\theta$, so $du = \cos\theta \, d\theta$. The limits change from 0 to 1:

$$\begin{aligned} J &= \frac{\rho_\gamma c}{4\pi}(2\pi) \int_0^1 u \, du \\ &= \frac{\rho_\gamma c}{2} \left[\frac{u^2}{2} \right]_0^1 \\ &= \frac{\rho_\gamma c}{4} \end{aligned}$$

Substitute our earlier result for the total energy density $\rho_\gamma = \frac{\pi^2 (k_B T)^4}{15(\hbar c)^3}$:

$$J = \frac{c}{4} \left(\frac{\pi^2 k_B^4}{15\hbar^3 c^3} \right) T^4 = \sigma T^4 \tag{8}$$

where σ is the **Stefan-Boltzmann constant**:

$$\boxed{\sigma \equiv \frac{\pi^2 k_B^4}{60\hbar^3 c^2} \approx 5.67 \times 10^{-8} \frac{\mathrm{W}}{\mathrm{m^2 K^4}}} \tag{9}$$

This is the Stefan-Boltzmann Law, which states that the total power radiated per unit area of a blackbody is proportional to the fourth power of its thermodynamic temperature.

3 The Ultraviolet Catastrophe and Quantum Mechanics

In deriving the Planck function, we assumed that photons come in discrete energy packets (quanta). This fundamentally assumes the rules of Quantum Mechanics.

For a single mode of frequency ω, the allowed energies are $E_n = n\hbar\omega$. The partition function is a discrete sum over these states:

$$Z_\omega = 1 + e^{-\beta\hbar\omega} + e^{-2\beta\hbar\omega} + \cdots = \frac{1}{1 - e^{-\beta\hbar\omega}} \tag{10}$$

Planck used this mathematical trick in 1900 to successfully derive the blackbody spectrum. He knew the answer was empirically right, but at the time, he did not emphasize that light physically exists as discrete quanta (a concept Einstein would later champion).

Without Quantum Mechanics, classical physics assumes that energy can take any continuous value. Applying classical equipartition to the electromagnetic cavity modes yields the Rayleigh-Jeans law. While the Rayleigh-Jeans law works well at low frequencies, it predicts that the energy density diverges to infinity at high frequencies (short wavelengths). This catastrophic failure of classical physics is known as the **Ultraviolet Catastrophe**. The assumption of quantized energy levels ($\hbar\omega$) is what mathematically suppresses the high-frequency modes and resolves the catastrophe.