Fermi Degeneracy Pressure and White Dwarfs

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1 Kinetic Energy of a Degenerate Fermi Gas

Consider a non-relativistic, fully degenerate Fermi gas at absolute zero ($T = 0$).

At $T = 0$, the Fermi-Dirac distribution $f_{FD}(E)$ is a sharp step function:

- All energy states below the Fermi energy are completely occupied ($f_{FD} = 1$).
- All energy states above the Fermi energy are completely empty ($f_{FD} = 0$).

This sharp cutoff defines the chemical potential at absolute zero: $\mu(T = 0) = E_F$.

The total average kinetic energy $\langle E \rangle$ of the gas is the sum of the energies of all occupied states:

$$\langle E \rangle = \sum_r \frac{E_r}{e^{\beta(E_r - \mu)} + 1} = \int_0^\infty \frac{E\, g(E)}{e^{\beta(E-\mu)} + 1} dE \tag{1}$$

Because we are at $T = 0$, the distribution function truncates the integral exactly at E_F:

$$\langle E \rangle = \int_0^{E_F} E\, g(E) dE \tag{2}$$

Recall the density of states $g(E)$ for a non-relativistic 3D gas:

$$g(E) = g_s \frac{4\pi\sqrt{2} m^{3/2} V}{h^3} \sqrt{E} \tag{3}$$

where g_s is the spin degeneracy factor (for electrons, $g_s = 2$).

Substituting $g(E)$ into the integral:

$$\begin{aligned}
\langle E \rangle &= \int_0^{E_F} E \left(g_s \frac{4\pi\sqrt{2} m^{3/2} V}{h^3} E^{1/2} \right) dE \\
&= g_s \frac{4\pi\sqrt{2} m^{3/2} V}{h^3} \int_0^{E_F} E^{3/2} dE \\
&= g_s \frac{4\pi\sqrt{2} m^{3/2} V}{h^3} \left[\frac{2}{5} E^{5/2} \right]_0^{E_F} \\
&= g_s \frac{4\pi\sqrt{2} m^{3/2} V}{h^3} \left(\frac{2}{5} E_F^{5/2} \right)
\end{aligned} \tag{4}$$

To simplify this expression, recall that we previously computed the total number of particles $\langle N \rangle$ by integrating the density of states up to E_F:

$$\langle N \rangle = \int_0^{E_F} g(E) dE = g_s \frac{4\pi\sqrt{2} m^{3/2} V}{h^3} \left(\frac{2}{3} E_F^{3/2} \right) \tag{5}$$

We can isolate the complicated prefactor from the $\langle N \rangle$ equation:

$$g_s \frac{4\pi\sqrt{2}m^{3/2}V}{h^3} = \frac{3\langle N \rangle}{2E_F^{3/2}} \tag{6}$$

Substituting this relation back into our equation for the total energy:

$$\begin{aligned}\langle E \rangle &= \left(\frac{3\langle N \rangle}{2E_F^{3/2}}\right)\left(\frac{2}{5}E_F^{5/2}\right) \\ &= \frac{6}{10}\langle N \rangle E_F^{(5/2-3/2)} \end{aligned} \tag{7}$$

$$\boxed{\langle E \rangle = \frac{3}{5}\langle N \rangle E_F} \tag{8}$$

Thus, the average kinetic energy per particle in a fully degenerate Fermi gas is exactly $\frac{3}{5}E_F$.

2 Fermi Degeneracy Pressure

Even at absolute zero, a Fermi gas exerts a pressure. This is a purely quantum mechanical effect arising from the Pauli Exclusion Principle, which forces particles into higher energy/momentum states, creating an outward macroscopic pressure known as **degeneracy pressure**.

We can compute this pressure using the fundamental thermodynamic identity at constant entropy and particle number ($dS = 0, dN = 0$):

$$P = -\left(\frac{\partial\langle E \rangle}{\partial V}\right)_{S,N} \tag{9}$$

Substitute our result for the total energy:

$$P = -\frac{\partial}{\partial V}\left(\frac{3}{5}\langle N \rangle E_F\right) = -\frac{3}{5}\langle N \rangle \frac{\partial E_F}{\partial V} \tag{10}$$

To evaluate the derivative, recall our expression for the Fermi energy:

$$E_F = \frac{h^2}{2m}\left(\frac{3N}{4\pi V g_s}\right)^{2/3} \tag{11}$$

Notice that the Fermi energy is proportional to $V^{-2/3}$. Let us define all the volume-independent terms as a constant C:

$$E_F = CV^{-2/3} \tag{12}$$

Taking the volume derivative yields:

$$\begin{aligned}\frac{\partial E_F}{\partial V} &= -\frac{2}{3}CV^{-5/3} \\ &= -\frac{2}{3}\left(CV^{-2/3}\right)V^{-1} \\ &= -\frac{2}{3}\frac{E_F}{V} \end{aligned} \tag{13}$$

Finally, substitute this volume derivative back into our pressure equation:

$$
\begin{aligned}
P &= -\frac{3}{5}\langle N\rangle\left(-\frac{2}{3}\frac{E_F}{V}\right) \\
&= \frac{6}{15}\frac{\langle N\rangle}{V}E_F
\end{aligned} \tag{14}
$$

$$
\boxed{P = \frac{2}{5}\frac{\langle N\rangle}{V}E_F} \tag{15}
$$

This is the electron degeneracy pressure. It is precisely this quantum pressure that supports the mass of a **White Dwarf** star against the crushing force of its own gravitational collapse.

3 White Dwarfs

White dwarfs are stellar remnant held against gravitational collapse by Fermi degeneracy pressure. They are the end point of low-mass stars that have exhausted their hydrogen fuel. Here, we would like to compute the maximum mass (referred to as the *Chandrasekhar mass*) that a white dwarf (WD) can have. We will make the following modeling choices:

- Model the WD as a uniform sphere of radius R made of N electrons and N protons (to ensure charge neutrality).
- Since the Fermi energy is $E_F \propto 1/m$ (m being the mass of the fermion), we can neglect the proton contribution to the degeneracy pressure of the star.
- Since $m_p \simeq 1836 m_e$, the mass M of the WD is dominated by the protons with $M \simeq N m_p$.
- The gravitation binding energy of the star is

$$
E_{\text{grav}} = -\frac{3}{5}\frac{G_N M^2}{R}. \tag{16}
$$

- This gravitational binding energy must be balanced by the kinetic energy of the degenerate electrons.
- While WDs are technically hot, we can consider the degenerate electrons to be at $T = 0$. This is a good approximation since the Fermi energy of the electrons is $E_F \gg T_{\text{WD}}$, the actual temperature of the WD.
- We will take the electrons to be ultra-relativistic with $E = pc$.

In the worksheet, we will use the modeling choices to compute the Chandrasekhar mass.