

# PHYS 301

## Thermodynamics and Statistical Mechanics

Problems #10  
Wednesday, 04/01/2026

### Question 1.

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We know that an ideal gas has an equation of state given by  $PV = Nk_B T$ . This purely classical result gets corrected by quantum effects, which depends on whether the particles involved are bosons or fermions. Here, we want to compute the first order correction to the ideal gas law for a gas of non-relativistic bosons, valid at leading order in  $\lambda_Q^3 N/V \ll 1$ , where  $\lambda_Q$  is the thermal de Broglie wavelength of the particles.

The grand partition function for a gas of bosons of mass  $m$  at temperature  $T$  and chemical potential  $\mu < 0$  is

$$\mathcal{Z} = \prod_i \frac{1}{1 - e^{-\beta(E_i - \mu)}}, \quad (1)$$

where the  $E_i$ s are the different energy levels that the bosons can occupy, and the product is over all possible energy levels. From this grand partition function, we can define the grand potential  $\Phi = -k_B T \ln \mathcal{Z}$ . This potential is super useful since you may remember that  $\Phi = -PV$ , where  $P$  is the pressure of the gas and  $V$  is its volume. This means that the pressure of this boson gas is

$$PV = k_B T \ln \mathcal{Z}. \quad (2)$$

Using the property of logarithms, we can write this as

$$\begin{aligned} PV &= k_B T \ln \left( \prod_i \frac{1}{1 - e^{-\beta(E_i - \mu)}} \right) \\ &= -k_B T \sum_i \ln \left( 1 - e^{-\beta(E_i - \mu)} \right) \\ &= -k_B T \int dE g(E) \ln \left( 1 - e^{-\beta(E - \mu)} \right), \end{aligned} \quad (3)$$

where  $g(E)$  is the density of states (the number of states between energy  $E$  and  $E + dE$ ), which for non-relativistic bosons is

$$g(E) = \frac{4\pi\sqrt{2} V m^{3/2}}{h^3} \sqrt{E} \quad (4)$$

(a) Use integration by parts to show that

$$PV = \frac{2}{3} \langle E \rangle, \quad (5)$$

where the average energy is

$$\langle E \rangle = \int dE \frac{g(E) E}{e^{\beta(E - \mu)} - 1}. \quad (6)$$

Now the issue is that  $\langle E \rangle$  depends on  $\mu$ , which is itself a function of temperature. To compute the right-hand side, we thus need to determine what  $\mu(T)$  is, or equivalently, what the fugacity  $z \equiv e^{\beta\mu}$  is. We will use the number of particles to determine  $z$ , in the limit that  $z \ll 1$ .

- (b) For a boson gas, we saw in class, we saw that the number of particles can be written as

$$N = \frac{V}{\lambda_Q^3} g_{3/2}(z), \quad (7)$$

where

$$g_{3/2}(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^{3/2}}. \quad (8)$$

Express  $N$  in the limit  $z \ll 1$ , up to quadratic order in  $z$ . Then solve this expression for  $z$  in the limit that  $\lambda_Q^3 N/V \ll 1$ . You should get

$$z = \frac{\lambda_Q^3 N}{V} \left( 1 - \frac{1}{2\sqrt{2}} \frac{\lambda_Q^3 N}{V} + \dots \right). \quad (9)$$

- (c) Using Eq. (6) above, Taylor expand  $\langle E \rangle$  to quadratic order in  $z$ , in the limit that  $z \ll 1$ . Then use Eq. (5) above together with the result from part (b) to show that

$$PV = Nk_B T \left( 1 - \frac{\lambda_Q^3 N}{4\sqrt{2}V} + \dots \right), \quad (10)$$

which shows that a bosonic gas has slightly less pressure than an ideal gas. This is entirely caused by quantum statistics: bosons like to be on top of each other (an effect called *Bose enhancement*) and so they do not offer as much resistance when we compress them.