PHYS 301: Thermodynamics and Statistical Mechanics
Solutions to Problem Set #11

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Question 1: Ultra-Relativistic Electron Gas

We are considering a gas of non-interacting, ultra-relativistic electrons. For ultra-relativistic particles, the rest mass is negligible, and the energy-momentum relationship is:

$$E = pc \implies p = \frac{E}{c} \quad \text{and} \quad dp = \frac{dE}{c} \tag{1}$$

Electrons are fermions with a spin degeneracy of $g_s = 2$.

First, we find the density of states $g(E)$. In a volume V, the number of states in a momentum shell dp is:

$$g(p)\,dp = g_s \frac{V}{h^3} 4\pi p^2\,dp \tag{2}$$

Substituting the energy relations into this expression:

$$g(E)\,dE = 2\frac{V}{h^3} 4\pi \left(\frac{E}{c}\right)^2 \left(\frac{dE}{c}\right) = \frac{8\pi V}{h^3 c^3} E^2\,dE \tag{3}$$

(a) Integral for the Grand Canonical Potential

The grand partition function for a gas of fermions is:

$$\mathcal{Z} = \prod_i \left(1 + e^{-\beta(E_i - \mu)}\right) \tag{4}$$

The grand canonical potential is defined as $\Phi = -k_B T \ln \mathcal{Z}$. Taking the logarithm converts the product into a sum over all states:

$$\Phi = -k_B T \sum_i \ln\left(1 + e^{-\beta(E_i - \mu)}\right) \tag{5}$$

Converting this sum into an integral over the continuous density of states $g(E)$ derived above:

$$\Phi = -k_B T \int_0^\infty dE\, g(E) \ln\left(1 + e^{-\beta(E - \mu)}\right) \tag{6}$$

$$\boxed{\Phi = -k_B T \frac{8\pi V}{h^3 c^3} \int_0^\infty dE\, E^2 \ln\left(1 + e^{-\beta(E - \mu)}\right)} \tag{7}$$

(b) Show that $PV = \langle E \rangle / 3$

We know from thermodynamics that $\Phi = -PV$. Therefore:

$$PV = k_B T \int_0^\infty dE\, g(E) \ln\left(1 + e^{-\beta(E-\mu)}\right) \tag{8}$$

We evaluate this integral using integration by parts ($\int u\, dv = uv - \int v\, du$). Let:

$$u = \ln\left(1 + e^{-\beta(E-\mu)}\right)$$
$$dv = g(E)\, dE = \frac{8\pi V}{h^3 c^3} E^2\, dE$$

Taking the derivative of u and the integral of dv:

$$du = \frac{1}{1 + e^{-\beta(E-\mu)}} \left(-\beta e^{-\beta(E-\mu)}\right) dE = \frac{-\beta}{e^{\beta(E-\mu)} + 1} dE$$
$$v = \int \frac{8\pi V}{h^3 c^3} E^2\, dE = \frac{8\pi V}{h^3 c^3} \frac{E^3}{3} = \frac{1}{3} E g(E)$$

The boundary term $[uv]_0^\infty$ vanishes: at $E = 0$, $v = 0$ because of the E^3 factor; at $E \to \infty$, $u \to 0$ exponentially, which dominates the polynomial growth of v. Thus:

$$\begin{aligned} PV &= k_B T \left(0 - \int_0^\infty v\, du\right) \\ &= -k_B T \int_0^\infty \left(\frac{1}{3} E g(E)\right) \left(\frac{-\beta}{e^{\beta(E-\mu)} + 1}\right) dE \end{aligned} \tag{9}$$

Since $k_B T \beta = 1$, the constants cancel out, leaving:

$$PV = \frac{1}{3} \int_0^\infty dE\, \frac{E g(E)}{e^{\beta(E-\mu)} + 1} \tag{10}$$

Recognizing the integral on the right as the definition of the average total energy $\langle E \rangle = \int E g(E) f(E) dE$, we get:

$$\boxed{PV = \frac{1}{3} \langle E \rangle} \tag{11}$$

(c) Show that at zero temperature $PV^{4/3} = \textbf{const}$

At absolute zero ($T = 0$), the Fermi-Dirac distribution becomes a sharp step function. All states are filled up to the Fermi energy E_F, and all states above are empty.

First, let's find the total number of particles N at $T = 0$:

$$\begin{aligned} N &= \int_0^{E_F} g(E)\, dE = \frac{8\pi V}{h^3 c^3} \int_0^{E_F} E^2\, dE \\ &= \frac{8\pi V}{h^3 c^3} \frac{E_F^3}{3} \end{aligned} \tag{12}$$

We can invert this to find the Fermi energy as a function of the particle density:

$$E_F = \left(\frac{3 N h^3 c^3}{8\pi V}\right)^{1/3} \propto V^{-1/3} \tag{13}$$

Next, we calculate the total internal energy $\langle E \rangle$ at $T = 0$:

$$\begin{aligned} \langle E \rangle &= \int_0^{E_F} E g(E) \, dE = \frac{8\pi V}{h^3 c^3} \int_0^{E_F} E^3 \, dE \\ &= \frac{8\pi V}{h^3 c^3} \frac{E_F^4}{4} \end{aligned} \tag{14}$$

Notice that we can rewrite this in terms of N:

$$\langle E \rangle = \frac{3}{4} \left(\frac{8\pi V}{h^3 c^3} \frac{E_F^3}{3} \right) E_F = \frac{3}{4} N E_F \tag{15}$$

Now, substitute the expression for E_F into the energy equation:

$$\langle E \rangle = \frac{3}{4} N \left(\frac{3 N h^3 c^3}{8\pi V} \right)^{1/3} = \left[\frac{3}{4} N \left(\frac{3 N h^3 c^3}{8\pi} \right)^{1/3} \right] V^{-1/3} \tag{16}$$

For a closed system, the total number of particles N is constant. Therefore, the entire term in the brackets is a constant, let's call it K.

$$\langle E \rangle = K V^{-1/3} \tag{17}$$

Using the result from part (b), $P = \frac{\langle E \rangle}{3V}$:

$$P = \frac{1}{3V} \left(K V^{-1/3} \right) = \frac{K}{3} V^{-4/3} \tag{18}$$

Rearranging this equation yields:

$$\boxed{P V^{4/3} = \text{const}} \tag{19}$$

This is the equation of state for an adiabatic expansion of an ultra-relativistic Fermi gas, such as the degenerate electron gas supporting a white dwarf star against gravitational collapse.