

PHYS 301

Thermodynamics and Statistical Mechanics

Problems #12
Wednesday, 04/15/2026

Question 1.

Consider a gas of N of *hard* discs of radius $r_0/2$ and mass m in two dimensions. The temperature is T and the disks are located in an area A . The goal of this question is to compute the equation of state for this 2D gas, in the limit that $A/N \gg r_0^2$ and keeping the first-order correction to the ideal gas case.

(a) Compute the partition function Z for this gas using

$$Z(N, A, T) = \frac{1}{N! h^2} \int \prod_{i=1}^N d^2 p_i d^2 x_i e^{-\beta H}, \quad (1)$$

where H is the (classical) Hamiltonian of the system

$$H = \sum_{i=1}^N \frac{p_i^2}{2m} + \sum_{i>j} U(r_{ij}), \quad (2)$$

where p_i is the momentum of the i th discs and $U(r_{ij})$ is the potential energy between disc i and disc j , with $r_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$. This potential has the form

$$U(r) = \begin{cases} \infty & \text{for } r < r_0 \\ 0 & \text{for } r \geq r_0 \end{cases}, \quad (3)$$

which characterizes what we mean by hard discs. You should first convince yourself that the partition reduces to this form

$$Z(N, A, T) = \frac{A^N}{N! \lambda_Q^{2N}} \left(1 + \frac{N}{2A} \int d^2 r f(r) + \dots \right)^N, \quad (4)$$

where $f(r) = e^{-\beta U(r)} - 1$, and then compute the integral.

(b) Compute the Helmholtz free energy $F = -k_B T \ln Z$ for this gas.

(c) Using the standard relation,

$$P = -\frac{\partial F}{\partial A}, \quad (5)$$

compute the equation of state for this gas of hard discs.