PHYS 301: Thermodynamics and Statistical Mechanics
Solutions to Problem Set #1

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January 21, 2026

Question 1: The Two-State System (Coin Flips)

Problem Statement: Consider a system of $N = 50$ fair coins.

(a) **How many possible microstates are there? How many macrostates?**

The number of microstates for a system of N particles with 2 degrees of freedom is given by $\Omega_{\text{total}} = 2^N$.

$$\Omega_{\text{total}} = 2^{50} \approx 1.126 \times 10^{15} \tag{1}$$

A macrostate is defined by the total number of heads, n. Since n can range from 0 (all tails) to 50 (all heads), the number of macrostates is:

$$N + 1 = 50 + 1 = 51 \text{ macrostates} \tag{2}$$

(b) **How many ways are there of getting exactly 25 heads and 25 tails?**

The multiplicity $\Omega(n)$ of getting exactly n heads is given by the binomial coefficient:

$$\Omega(N, n) = \binom{N}{n} = \frac{N!}{n!(N-n)!} \tag{3}$$

For $N = 50$ and $n = 25$:

$$\Omega(50, 25) = \frac{50!}{25!25!} \approx 1.264 \times 10^{14} \tag{4}$$

(c) **What is the probability of getting exactly 25 heads and 25 tails?**

The probability $P(n)$ is the multiplicity of the macrostate divided by the total number of microstates:

$$P(25) = \frac{\Omega(50, 25)}{\Omega_{\text{total}}} = \frac{1.264 \times 10^{14}}{1.126 \times 10^{15}} \approx 0.112 \quad (11.2\%) \tag{5}$$

(d) **What is the probability of getting exactly 30 heads and 20 tails?**

For $n = 30$:

$$P(30) = \frac{50!}{30!20! \cdot 2^{50}} = \frac{4.713 \times 10^{13}}{1.126 \times 10^{15}} \approx 0.042 \quad (4.2\%) \tag{6}$$

(e) **What is the probability of getting exactly 40 heads and 10 tails?**

For $n = 40$:

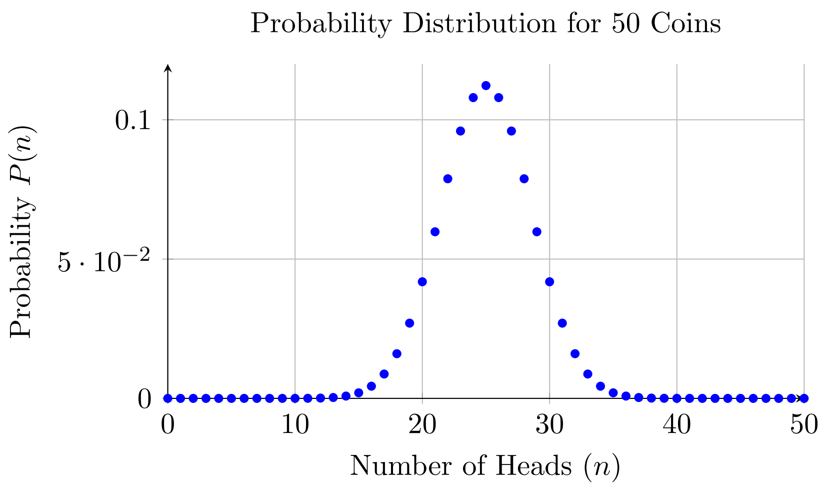
$$P(40) = \frac{50!}{40!10! \cdot 2^{50}} = \frac{1.027 \times 10^{10}}{1.126 \times 10^{15}} \approx 9.12 \times 10^{-6} \quad (0.0009\%) \tag{7}$$

(f) **What is the probability of getting exactly 50 heads?**

For $n = 50$ (only 1 microstate):

$$P(50) = \frac{1}{2^{50}} \approx 8.88 \times 10^{-16} \tag{8}$$

(g) **Sketch a graph of the probability of getting n heads, as a function of n.**



Note: The distribution is sharply peaked around the mean value $n = 25$.

Question 2: The Einstein Solid

Problem Statement: An Einstein solid consists of N independent quantum harmonic oscillators. The energy of the system is characterized by the total number of energy quanta, $q = \sum n_i$.

(a) **List all possible microstates for $N = 3$ oscillators sharing $q = 2$ energy quanta.**

We require integer solutions to $n_1 + n_2 + n_3 = 2$. The microstates (n_1, n_2, n_3) are:

- (2, 0, 0)
- (0, 2, 0)
- (0, 0, 2)
- (1, 1, 0)
- (1, 0, 1)
- (0, 1, 1)

Total Multiplicity $\Omega = 6$.

(b) **List all possible microstates for $N = 3$ oscillators sharing $q = 3$ energy quanta.**

We require integer solutions to $n_1 + n_2 + n_3 = 3$.

- Permutations of (3, 0, 0): (3,0,0), (0,3,0), (0,0,3) $\rightarrow$ 3 states
- Permutations of (2, 1, 0): (2,1,0), (2,0,1), (1,2,0), (0,2,1), (1,0,2), (0,1,2) $\rightarrow$ 6 states
- Permutations of (1, 1, 1): (1,1,1) $\rightarrow$ 1 state

Total Multiplicity $\Omega = 3 + 6 + 1 = 10$.

(c) **List all possible microstates for $N = 3$ oscillators sharing $q = 4$ energy quanta.**

We require integer solutions to $n_1 + n_2 + n_3 = 4$.

- Permutations of (4, 0, 0) $\rightarrow$ 3 states
- Permutations of (3, 1, 0) $\rightarrow$ 6 states
- Permutations of (2, 2, 0) $\rightarrow$ 3 states
- Permutations of (2, 1, 1) $\rightarrow$ 3 states

Total Multiplicity $\Omega = 3 + 6 + 3 + 3 = 15$.

(d) **Argue that for N oscillators sharing q energy quanta, the multiplicity is:**

$$\Omega(N, q) = \frac{(q + N - 1)!}{q!(N - 1)!}. \tag{9}$$

Solution:

This problem can be mapped to a combinatorial "Stars and Bars" problem .

- We have q units of energy (quanta), which we can visualize as indistinguishable items (stars).
- We have N distinct oscillators (bins). To separate the quanta into N bins, we need $N-1$ dividers (bars).

Consider a linear arrangement of these items. The total number of positions in the line is the sum of the number of stars and bars:

$$\text{Total positions} = q + (N - 1)$$

We need to choose which of these positions are occupied by the q energy quanta (the stars). Once the positions of the quanta are chosen, the positions of the dividers (bars) are automatically determined.

The number of ways to choose q positions out of a total of $q + N - 1$ is given by the binomial coefficient:

$$\binom{q + N - 1}{q}$$

Expanding this using the definition of the binomial coefficient:

$$\Omega(N, q) = \frac{(q + N - 1)!}{q!((q + N - 1) - q)!} = \frac{(q + N - 1)!}{q!(N - 1)!}$$

This matches the required formula.