PHYS 301: Thermodynamics and Statistical Mechanics
Solutions to Problem Set #2

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Question 1: Einstein Solid in the Low-Temperature Limit

Problem: Derive an approximation for the multiplicity $\Omega(N, q)$ of an Einstein solid where $q \ll N$ (low energy limit). Assume N and q are large.

Solution:

We start with the general expression for the multiplicity of an Einstein solid:

$$\Omega(N, q) = \frac{(q + N - 1)!}{q!(N - 1)!} \tag{1}$$

Since N is large ($N \gg 1$), we can approximate $N - 1 \approx N$ and $q + N - 1 \approx q + N$. The expression becomes:

$$\Omega(N, q) \approx \frac{(q + N)!}{q!N!} \tag{2}$$

Next, we take the natural logarithm of the multiplicity to make use of Stirling's approximation ($\ln n! \approx n \ln n - n$):

$$\ln \Omega \approx \ln((q + N)!) - \ln(q!) - \ln(N!) \tag{3}$$

Applying Stirling's approximation:

$$\begin{aligned} \ln \Omega &\approx [(q + N) \ln(q + N) - (q + N)] - [q \ln q - q] - [N \ln N - N] \\ &= (q + N) \ln(q + N) - q - N - q \ln q + q - N \ln N + N \end{aligned}$$

The linear terms $(-q - N + q + N)$ cancel out:

$$\ln \Omega \approx (q + N) \ln(q + N) - q \ln q - N \ln N \tag{4}$$

We can factor out N from the logarithm in the first term: $\ln(q + N) = \ln(N(1 + q/N)) = \ln N + \ln(1 + q/N)$.

$$\ln \Omega \approx (q + N) \left[\ln N + \ln\left(1 + \frac{q}{N}\right)\right] - q \ln q - N \ln N \tag{5}$$

Since we are in the limit $q \ll N$, the ratio q/N is small. We use the Taylor series expansion $\ln(1 + x) \approx x$ for small x:

$$\ln\left(1 + \frac{q}{N}\right) \approx \frac{q}{N} \tag{6}$$

Substituting this back into the equation:

$$\begin{aligned} \ln \Omega &\approx (q + N) \left[\ln N + \frac{q}{N}\right] - q \ln q - N \ln N \\ &= q \ln N + \frac{q^2}{N} + N \ln N + q - q \ln q - N \ln N \end{aligned}$$

Canceling the $N \ln N$ terms:

$$\ln \Omega \approx q \ln N + \frac{q^2}{N} + q - q \ln q \tag{7}$$

Because $q \ll N$, the term $\frac{q^2}{N}$ is negligible compared to q. We drop it:

$$\begin{aligned} \ln \Omega &\approx q \ln N + q - q \ln q \\ &= q(\ln N + 1 - \ln q) \end{aligned}$$

Recalling that $1 = \ln e$, we can combine the terms inside the logarithm:

$$\ln \Omega \approx q(\ln N + \ln e - \ln q) = q \ln \left(\frac{Ne}{q} \right) \tag{8}$$

Exponentiating both sides gives the final expression for the multiplicity:

$$\boxed{\Omega(N, q) \approx \left(\frac{Ne}{q} \right)^q} \tag{9}$$

Question 2: Two-State Paramagnet in the High-Field Limit

Problem: Derive the multiplicity $\Omega(N, N_\downarrow)$ for a spin-1/2 system where $N_\downarrow \ll N$.

Solution:

We start with the general multiplicity for a system of N spins:

$$\Omega(N, N_\uparrow) = \frac{N!}{N_\uparrow! N_\downarrow!} \tag{10}$$

We are given the relation $N = N_\uparrow + N_\downarrow$. Therefore, $N_\uparrow = N - N_\downarrow$. Substituting this into the multiplicity equation:

$$\Omega = \frac{N!}{(N - N_\downarrow)! N_\downarrow!} \tag{11}$$

Taking the natural logarithm and applying Stirling's approximation:

$$\begin{aligned}
\ln \Omega &= \ln N! - \ln((N - N_\downarrow)!) - \ln N_\downarrow! \\
&\approx (N \ln N - N) - [(N - N_\downarrow) \ln(N - N_\downarrow) - (N - N_\downarrow)] - (N_\downarrow \ln N_\downarrow - N_\downarrow)
\end{aligned}$$

The linear terms are $-N - [-(N - N_\downarrow)] - [-N_\downarrow] = -N + N - N_\downarrow + N_\downarrow = 0$.

$$\ln \Omega \approx N \ln N - (N - N_\downarrow) \ln(N - N_\downarrow) - N_\downarrow \ln N_\downarrow \tag{12}$$

We expand the term $\ln(N - N_\downarrow)$ assuming $N_\downarrow \ll N$:

$$\ln(N - N_\downarrow) = \ln \left[N \left(1 - \frac{N_\downarrow}{N} \right) \right] = \ln N + \ln \left(1 - \frac{N_\downarrow}{N} \right) \approx \ln N - \frac{N_\downarrow}{N} \tag{13}$$

Substitute this approximation back into the expression for $\ln \Omega$:

$$\begin{aligned}
\ln \Omega &\approx N \ln N - (N - N_\downarrow) \left(\ln N - \frac{N_\downarrow}{N} \right) - N_\downarrow \ln N_\downarrow \\
&= N \ln N - \left[N \ln N - N_\downarrow - N_\downarrow \ln N + \frac{N_\downarrow^2}{N} \right] - N_\downarrow \ln N_\downarrow \\
&= N \ln N - N \ln N + N_\downarrow + N_\downarrow \ln N - \frac{N_\downarrow^2}{N} - N_\downarrow \ln N_\downarrow
\end{aligned}$$

Canceling terms and dropping $\frac{N_\downarrow^2}{N}$ (since $N_\downarrow \ll N$ implies $\frac{N_\downarrow^2}{N} \ll N_\downarrow$):

$$\begin{aligned}
\ln \Omega &\approx N_\downarrow + N_\downarrow \ln N - N_\downarrow \ln N_\downarrow \\
&= N_\downarrow (1 + \ln N - \ln N_\downarrow) \\
&= N_\downarrow (\ln e + \ln N - \ln N_\downarrow) \\
&= N_\downarrow \ln \left(\frac{Ne}{N_\downarrow} \right)
\end{aligned}$$

Exponentiating both sides:

$$\boxed{\Omega(N, N_\downarrow) \approx \left(\frac{Ne}{N_\downarrow} \right)^{N_\downarrow}} \tag{14}$$

Comparison

Comparing the results from Question 1 and Question 2:

- Einstein Solid ($q \ll N$): $\Omega \approx \left(\frac{Ne}{q}\right)^q$
- Two-State System ($N_\downarrow \ll N$): $\Omega \approx \left(\frac{Ne}{N_\downarrow}\right)^{N_\downarrow}$

The expressions are identical in form, with the number of energy quanta q in the Einstein solid playing the same role as the number of down spins $N_\downarrow$ in the paramagnet.

This similarity arises because in the limit $q \ll N$, the likelihood of putting more than one quantum of energy into a single oscillator is negligible. The quanta behave like a dilute gas of indistinguishable particles occupying N distinguishable sites. Similarly, when $N_\downarrow \ll N$, the "down spins" are sparse and act like a dilute gas of particles distributed among the N lattice sites. In both "dilute" limits, the statistics simplify to the same form because the exclusion principle (relevant for spins, as you can't have two down spins on one site) becomes irrelevant when the "particles" are so sparse they rarely try to occupy the same site anyway.