PHYS 301
Thermodynamics and Statistical Mechanics

Problems #4

Wednesday, 02/11/2026

Question 1.

In class, we derived the Boltzmann distribution for the probability of finding a system S in a specific microstate $|n\rangle$ with energy E_n when it is in thermal contact with a reservoir at temperature T

$$p(n) = \frac{e^{-E_n/k_B T}}{\sum_m e^{-E_m/k_B T}}. \tag{1}$$

It turns out the normalization factor in the denominator is a very important quantity called the **partition function**, usually denoted by Z. Defining the inverse temperature $\beta \equiv 1/(k_\mathrm{B}T)$, this partition function can be written as

$$Z(\beta) = \sum_n e^{-\beta E_n} \tag{2}$$

Let's compute this partition function for a very simple system: a quantum harmonic oscillator. As we saw before, the energy level of such oscillators are

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega \quad n = 0, 1, 2, \ldots \tag{3}$$

(a) Compute the partition function Z for the quantum harmonic oscillator. You may find the following result useful

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \text{for } x < 1. \tag{4}$$

(b) Compute the average energy of the system at temperature T

$$\langle E \rangle = \sum_{n=0}^{\infty} E_n \, p(n) \tag{5}$$

(c) Now compute the quantity

$$-\frac{\partial}{\partial \beta} \ln Z. \tag{6}$$

Compare your answer to what you got in part (b). How general is this result? What is the relationship between $\langle E \rangle$ and Z?

Question 2.

A key property of the partition function is that, for two independent and distinguishable systems in thermal contact, each with energy levels $E_n^{(1)}$ and $E_m^{(2)}$, respectively, their partition functions multiply:

$$
\begin{aligned}
Z_{\text{tot}} &= \sum_{n,m} e^{-\beta(E_n^{(1)}+E_m^{(2)})} && (7)\\
&= \sum_{n,m} e^{-\beta E_n^{(1)}} e^{-\beta E_m^{(2)}} && (8)\\
&= \sum_{n} e^{-\beta E_n^{(1)}} \sum_{m} e^{-\beta E_m^{(2)}} && (9)\\
&= Z_1 Z_2 && (10)
\end{aligned}
$$

Each atom in a 3D Einstein solid can be seen as 3 independent quantum harmonic oscillators, one for each direction (x, y, z). Using a generalization of the above argument, compute the partition function of an Einstein solid with N atoms at temperature T. Each atom can be considered independent and distinguishable.