PHYS 301: Thermodynamics and Statistical Mechanics
Solutions to Problem Set #5

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Question 1: The Ultra-Relativistic Ideal Gas

Problem Statement: Consider an ultra-relativistic gas of N spinless particles obeying the energy-momentum relation $E = pc$, where c is the speed of light.

(a) Show that the canonical partition function is given by $Z(V,T) = \frac{1}{N!}[\frac{V}{\pi^2}(\frac{k_B T}{\hbar c})^3]^N$.

The partition function for N indistinguishable, non-interacting particles is given by:

$$Z_N = \frac{1}{N!}(Z_1)^N \tag{1}$$

where Z_1 is the single-particle partition function.

To find Z_1, we integrate over the classical phase space (position $\mathbf{r}$ and momentum $\mathbf{p}$), dividing by h^3 to account for the quantum volume of a state.

$$Z_1 = \frac{1}{h^3}\int d^3r \int d^3p\, e^{-\beta E(p)} \tag{2}$$

The spatial integral $\int d^3r$ simply yields the volume V. For the momentum integral, we switch to spherical coordinates ($d^3p = 4\pi p^2 dp$) and substitute the relativistic energy $E = pc$:

$$Z_1 = \frac{V}{h^3}\int_0^\infty 4\pi p^2 e^{-\beta pc}\, dp \tag{3}$$

Let $x = \beta cp$. Then $p = \frac{x}{\beta c}$ and $dp = \frac{dx}{\beta c}$. Substituting these into the integral:

$$\begin{aligned} Z_1 &= \frac{4\pi V}{h^3}\int_0^\infty \left(\frac{x}{\beta c}\right)^2 e^{-x}\frac{dx}{\beta c} \\ &= \frac{4\pi V}{h^3(\beta c)^3}\int_0^\infty x^2 e^{-x}\, dx \end{aligned} \tag{4}$$

The integral $\int_0^\infty x^2 e^{-x} dx = \Gamma(3) = 2! = 2$.

$$Z_1 = \frac{8\pi V}{h^3(\beta c)^3} \tag{5}$$

Now, we convert h to $\hbar$ using $h = 2\pi\hbar$:

$$Z_1 = \frac{8\pi V}{(2\pi\hbar)^3(\beta c)^3} = \frac{8\pi V}{8\pi^3\hbar^3(\beta c)^3} = \frac{V}{\pi^2(\beta\hbar c)^3} \tag{6}$$

Substituting $\beta = 1/k_B T$:

$$Z_1 = \frac{V}{\pi^2}\left(\frac{k_B T}{\hbar c}\right)^3 \tag{7}$$

Finally, for N particles:

$$\boxed{Z_N = \frac{1}{N!}\left[\frac{V}{\pi^2}\left(\frac{k_B T}{\hbar c}\right)^3\right]^N} \tag{8}$$

(b) Show that this gas obeys the ideal gas law $PV = Nk_BT$.

The pressure is defined thermodynamically as:

$$P = k_B T \left(\frac{\partial \ln Z_N}{\partial V}\right)_T \tag{9}$$

First, take the natural log of Z_N:

$$\begin{aligned}\ln Z_N &= \ln\left(\frac{1}{N!}\left[\frac{V}{\pi^2}\left(\frac{k_B T}{\hbar c}\right)^3\right]^N\right) \\ &= -\ln N! + N \ln\left[\frac{V}{\pi^2}\left(\frac{k_B T}{\hbar c}\right)^3\right] \\ &= -\ln N! + N\ln V + N \ln\left[\frac{1}{\pi^2}\left(\frac{k_B T}{\hbar c}\right)^3\right] \end{aligned} \tag{10}$$

Taking the derivative with respect to V:

$$\frac{\partial \ln Z_N}{\partial V} = \frac{\partial}{\partial V}(N \ln V) = \frac{N}{V} \tag{11}$$

Substitute this back into the pressure equation:

$$P = k_B T\left(\frac{N}{V}\right) \tag{12}$$

Rearranging gives the ideal gas law:

$$\boxed{PV = Nk_BT} \tag{13}$$

Note: This result implies that the equation of state depends only on the translational degrees of freedom and the non-interacting nature of the particles, not on the specific energy-momentum relation ($E = p^2/2m$ vs $E = pc$).

(c) Compute the average energy and heat capacity at temperature T.

The average energy is given by:

$$\langle E \rangle = -\frac{\partial \ln Z_N}{\partial \beta} \tag{14}$$

From part (a), we know $Z_N \propto \beta^{-3N}$. Let's isolate the β dependence in $\ln Z_N$:

$$\ln Z_N = \ln\left(\beta^{-3N}\right) + \text{terms independent of } \beta = -3N \ln \beta + \text{const} \tag{15}$$

Taking the derivative:

$$\langle E \rangle = -\frac{\partial}{\partial \beta}(-3N \ln \beta) = 3N\frac{1}{\beta} = 3Nk_BT \tag{16}$$

So, the average energy is:

$$\boxed{\langle E \rangle = 3Nk_BT} \tag{17}$$

Note: For a non-relativistic gas, $\langle E \rangle = \frac{3}{2}Nk_BT$. The factor of 3 comes from the linear dispersion relation.

The heat capacity at constant volume is:

$$C_V = \left(\frac{\partial \langle E \rangle}{\partial T}\right)_V = \frac{\partial}{\partial T}(3Nk_BT) \tag{18}$$

$$\boxed{C_V = 3Nk_B} \tag{19}$$

(d) Compute the entropy of this relativistic gas.

The entropy is given by $S = \frac{\langle E \rangle}{T} + k_B \ln Z_N$. First, use Stirling's approximation ($\ln N! \approx N \ln N - N$) to simplify $\ln Z_N$:

$$\begin{aligned} \ln Z_N &= N \ln \left[\frac{V}{\pi^2}\left(\frac{k_BT}{\hbar c}\right)^3\right] - (N \ln N - N) \\ &= N \ln \left[\frac{V}{N\pi^2}\left(\frac{k_BT}{\hbar c}\right)^3\right] + N \end{aligned} \tag{20}$$

Now substitute $\langle E \rangle = 3Nk_BT$ into the entropy formula:

$$\begin{aligned} S &= \frac{3Nk_BT}{T} + k_B \left(N \ln \left[\frac{V}{N\pi^2}\left(\frac{k_BT}{\hbar c}\right)^3\right] + N\right) \\ &= 3Nk_B + Nk_B \ln \left[\frac{V}{N\pi^2}\left(\frac{k_BT}{\hbar c}\right)^3\right] + Nk_B \\ &= 4Nk_B + Nk_B \ln \left[\frac{V}{N\pi^2}\left(\frac{k_BT}{\hbar c}\right)^3\right] \end{aligned} \tag{21}$$

Factoring out Nk_B:

$$\boxed{S = Nk_B \left[\ln \left(\frac{V}{N}\frac{1}{\pi^2}\left(\frac{k_BT}{\hbar c}\right)^3\right) + 4\right]} \tag{22}$$