PHYS 301
Thermodynamics and Statistical Mechanics

Worksheet #12

Tuesday March 31 2026

Question 1.

Bose-Einstein Condensation: Imagine we have a gas of non-relativistic indistinguishable bosons in a volume V and at temperature T. Each particle has mass m. Assume that the ground state energy as $E = 0$. Here we fix the total number of particles to be N, implying that the chemical potential is $\mu = \mu(N, T) < 0$. We just saw in class that the number of particles in this gas can be written as

$$N = \sum_r \frac{1}{e^{\beta(E_r - \mu)} - 1} = \frac{4\sqrt{2}\pi V m^{3/2}}{h^3} \int_0^\infty \frac{\sqrt{E} dE}{e^{\beta(E-\mu)} - 1} = \frac{V}{\lambda_Q^3} g_{3/2}(z), \tag{1}$$

where $\lambda_Q = h/\sqrt{2\pi m k_B T)}$ is the usual thermal de Broglie wavelength of the particles, $z \equiv e^{\beta\mu}$ is the fugacity, and

$$g_{3/2}(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^{3/2}}. \tag{2}$$

(a) Now imagine that we cool this system. To keep N fixed, the ration $g_{3/2}(z)/\lambda_Q^3$ must be kept constant. Since λ_Q increases as T goes down, $g_{3/2}(z)$ must also increase as the system is cooled. But since $\mu < 0$, $z \in (0, 1)$ and $g_{3/2}(z)$ can thus only increase so much. Compute the critical temperature T_c (as a function of N/V and m) at which the maximal value of $g_{3/2}(z)$ is reached.

(b) What happens at $T < T_c$? From Eq. (1) above, it seems that N must now decrease since $g_{3/2}(z)$ is saturated. But that's a stupid conclusion! N is fixed: I can't make particle disappear by cooling them down. Did we miss something when converting the discrete sum in Eq. (1) into the integral over the energy? *Hint: consider the ground state term with $E = 0$.*

(c) Show that the occupancy of the ground state (denoted here N_0) becomes of the order of N for $T < T_c$, with a scaling

$$\frac{N_0}{N} = 1 - \left(\frac{T}{T_c}\right)^{3/2}. \tag{3}$$

Thus, all N particles in the system lose their individual identities and become part of giant communist collective sitting in the ground state of the system. This is known as a *Bose-Einstein condensate.*