PHYS 301
Thermodynamics and Statistical Mechanics

Worksheet #14
Tuesday April 7 2026

Question 1.

Chandrasekhar Mass: White dwarfs are compact stars supported against gravitational collapse by electron degeneracy pressure. Since this pressure is finite, there is a critical maximum mass that a white dwarf can have before it collapses and then spectacularly explodes in a phenomenon called a type Ia supernova. Because this explosion always occurs at this mass, all these type Ia supernovae have the same intrinsic brightness and can thus be used to map the cosmos.

Here we will model a white dwarf as a uniform sphere of N free protons and N free electrons (so that the star is electrically neutral). We will model the free electrons as a $T = 0$ Fermi degenerate gas. The star has total mass M and radius R. Since the Fermi energy $E_F \propto 1/m$, where m is the mass of a single fermion, we can neglect the contribution of the protons to the degeneracy pressure as $m_p \simeq 1836\, m_e$. However, the mass M of the white dwarf is dominated by protons and we can write $M \simeq N m_p$. The gravitational binding energy associated with this mass is

$$E_{\text{grav}} = -\frac{3}{5}\frac{G_N M^2}{R}. \tag{1}$$

This binding energy must be balanced by the kinetic energy of the degenerate electron gas within the white dwarf. In fact, this kinetic energy is so large that electrons are typically relativistic within white dwarfs with $E = pc$, where p is the momentum.

(a) Using the usual expression for the particle number of fermions,

$$\langle N \rangle = \frac{g_s}{h^3}\int d^3x d^3p \frac{1}{e^{\beta(E-\mu)}+1}, \tag{2}$$

where $g_s = 2$ account for the two spin states of the electrons, show that the Fermi energy is

$$E_F = \left(\frac{3\langle N\rangle h^3 c^3}{8\pi V}\right)^{1/3}. \tag{3}$$

Remember that for a $T = 0\,(\beta = \infty)$ degenerate electron gas, all electron states up to the Fermi energy $E_F = \mu(T = 0)$ are occupied, and none above E_F are, effectively turning the Fermi-Dirac distribution into a step function.

(b) Compute the kinetic energy of the degenerate electron gas

$$\langle E_{\text{kin}} \rangle = \frac{g_s}{h^3}\int d^3x d^3p \frac{E}{e^{\beta(E-\mu)}+1} \tag{4}$$

using the same tricks as in part (a). You should get $\langle E_{\text{kin}}\rangle = (3/4)\langle N\rangle E_F$. Note that $\langle E_{\text{kin}}\rangle \propto 1/R$ just like E_{grav}.

(c) To be stable, a white dwarf must have $\langle E_{\text{kin}} \rangle + E_{\text{grav}} \geq 0$. Using $\langle N \rangle \simeq N = M/m_p$ and the volume of a sphere of radius R, compute the Chandrasekhar mass M_C at which white dwarfs become unstable and collapse. Express your answer in solar masses $M_\odot = 1.989 \times 10^{30}$ Kg. Note how the answer is independent of the radius of the white dwarf.