PHYS480/581 Cosmology
Density parameters and the Hubble constant

(Dated: September 14, 2022)

I. THE FRIEDMANN EQUATION IN TERMS OF DENSITY PARAMETERS

We have derived the Friedmann equation

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\sum_i \rho_i(t) - \frac{k}{a^2}, \tag{1}$$

where $\rho_i(t)$ is energy density in the ith component of the Universe, and where k is telling us about the spatial curvature of the Universe. Evaluating this equation at the present time (t_0) with the standard normalization that $a(t_0) = 1$, we obtain

$$H_0^2 = \frac{8\pi G}{3}\sum_i \rho_i(t_0) - k, \tag{2}$$

where we have introduced the Hubble constant $H_0 \equiv H(t_0)$. Slightly manipulating this yields

$$\frac{3H_0^2}{8\pi G} = \sum_i \rho_i(t_0) - \frac{3k}{8\pi G}. \tag{3}$$

The left-hand side of this equation is a very important quantity: it is the critical energy density $\rho_{\rm c}$ that the Universe needs to have (given a Hubble constant) for it to be spatially flat ($k = 0$). This critical density is

$$\rho_{\rm c} \equiv \frac{3H_0^2}{8\pi G}. \tag{4}$$

Since this critical density is so important for the evolution of our Universe, we like to refer to other energy densities (matter, radiation, etc.) in terms of their ratio to this critical density. We call these dimensionless ratios "density parameters" Ω_i, which are defined by

$$\Omega_i \equiv \frac{\rho_i(t_0)}{\rho_{\rm c}}. \tag{5}$$

Note that we always define the density parameters (in this class) at the present epoch. Other references sometime define their density parameters to be time-dependent, but this often confuses the picture. For us, the density parameters are constant describing how much each constituents of the Universe contribute to the critical density of the Universe. With this definition, we can then rewrite Eq. (3) as

$$1 = \sum_i \Omega_i + \Omega_K, \tag{6}$$

where we have identified the last term on the right-hand side of Eq. (3) with the "effective" energy density in curvature $\rho_K(t_0) = -3k/8\pi G$, and $\Omega_K = \rho_K(t_0)/\rho_{\rm c}$. This simply says that the sum over the density parameter of all the constituents of the Universe (including curvature) must sum up to 1. This is often written as

$$\Omega_K = 1 - \sum_i \Omega_i. \tag{7}$$

In a spatially flat universe ($\Omega_K = 0$), we must then have that $\sum_i \Omega_i = 1$. Now, going back to the general Friedmann equation (Eq. (1) above), we can use the definition of the critical density to rewrite is as

$$H^2 = H_0^2\left(\sum_i \frac{\rho_i(t)}{\rho_{\rm c}} + \frac{\rho_K}{a^2\rho_{\rm c}}\right). \tag{8}$$

To make further progress, we need to specify the kind of constituents populating our Universe: non-relativistic matter, radiation, something like a cosmological constant Λ (dark energy). These scale as

$$\rho_{\rm m}(t) = \rho_{\rm m}(t_0)a^{-3} \tag{9}$$

$$\rho_{\rm rad}(t) = \rho_{\rm rad}(t_0)a^{-4} \tag{10}$$

$$\rho_{\Lambda}(t) = \rho_{\Lambda}(t_0), \tag{11}$$

where we have used the standard convention $a(t_0) = 1$. Summing over these three components, we obtain

$$H^2 = H_0^2 \left(\Omega_{\rm rad}a^{-4} + \Omega_{\rm m}a^{-3} + \Omega_K a^{-2} + \Omega_\Lambda\right). \tag{12}$$

This form of the Friedmann equation is extremely useful as it is written in terms of quantities that are measured by observation. In our current Universe, it appears that $\Omega_{\rm m} \simeq 0.3$, $\Omega_\Lambda \simeq 0.7$, $\Omega_K \simeq 0$, and $\Omega_{\rm rad} \simeq 10^{-5}$.

We often write the Hubble constant H_0 as

$$H_0 = 100h\text{km/s/Mpc} = 1.023h \times 10^{-10}\,\text{year}^{-1} \tag{13}$$

where h is a dimensionless number (sometime called the reduced Hubble rate) introduced to capture the uncertainty in the actual value of the Hubble constant. While different h measurements seem to disagree, we roughly have $h \approx 0.7$.

II. AGE OF THE UNIVERSE

Given the values of the cosmological parameters, how do we determine the age of the Universe t_0? The rather bone-headed answer is

$$t_0 = \int_0^{t_0} dt. \tag{14}$$

The trick is to rewrite the small time interval dt using the Hubble expansion rate $H = \dot{a}/a$. Solving for dt here, we have

$$H = \frac{1}{a}\frac{da}{dt} \to dt = \frac{da}{aH}, \tag{15}$$

and the age of the Universe can then be written as

$$t_0 = \int_0^1 \frac{da}{aH(a)}. \tag{16}$$

Using Eq. (12), this can be written as

$$t_0 = \frac{1}{H_0}\int_0^1 \frac{da}{a\left[\Omega_{\rm rad}a^{-4} + \Omega_{\rm m}a^{-3} + \Omega_K a^{-2} + \Omega_\Lambda\right]^{1/2}}. \tag{17}$$

The fact that the age of the Universe is inversely proportional to the Hubble constant is very important. In fact, we call this time scale the "Hubble time"

$$\text{Hubble time} \equiv 1/H_0. \tag{18}$$

The Hubble time is basically the dynamical time scale of the Universe, or in other words, the time it takes for the Universe to change significantly. The Hubble time is

$$1/H_0 = 9.77h^{-1} \times 10^9\,\text{years}, \tag{19}$$

so, up to order-one factors, we know that the age of the Universe is of the order of ten billion years. In our Universe, with the values of the density parameters given above, a good approximation for the age of the Universe is

$$t_0 \simeq \frac{1}{H_0}\int_0^1 \frac{da}{a\left[\Omega_{\rm m}a^{-3} + \Omega_\Lambda\right]^{1/2}}. \tag{20}$$