PHYS480/581 Cosmology
Spatial Curvature

(Dated: September 12, 2022)

I. SPATIAL GEOMETRIES COMPATIBLE WITH THE COSMOLOGICAL PRINCIPLE.

Using the symmetries embedded in the cosmological principle (homogeneity and isotropy), we argued that the spacetime metric allowing us to measure distances in our Universe was

$$ds^2 = -dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right), \tag{1}$$

where $a(t)$ is the scale factor describing the expansion (or contraction) of the Universe. It turns out that this metric is a little too restrictive: there is a slightly more general metric that also obeys all the symmetries of the cosmological principle. To see this, it is most convenient to first rewrite the above metric using spherical coordinates

$$ds^2 = -dt^2 + a^2(t)\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]. \tag{2}$$

The above metric respects homogeneity and isotropy, but so would a metric of the form

$$ds^2 = -dt^2 + a^2(t)\left[f(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right], \tag{3}$$

since any function of r will respect isotropy. To also respect homogeneity, $f(r)$ needs to admit a specific form

$$f(r) = \frac{1}{1 - kr^2}, \tag{4}$$

where k is a constant with units of [length]$^{-2}$ which could have either signs. The case $k = 0$ is what we have been focusing on so far, but the cases with $k > 0$ and $k < 0$ bring in new spatial geometries that respect the cosmological principle. To understand these other geometries, it is informative to introduce a new radial coordinate via

$$d\chi \equiv \frac{dr}{\sqrt{1 - kr^2}}. \tag{5}$$

Using integration, this definition can be inverted for r. To do so, it is useful to write $k = \kappa/R^2$, where $\kappa = \{-1, 0, 1\}$, and R is a constant with units of [length] whose physical meaning will become clear soon. Then, defining $u = r/R$, we can write

$$\chi = R\int_0^{r/R}\frac{du}{\sqrt{1-\kappa u^2}} = \begin{cases} R\sinh^{-1}(r/R) & \text{if } \kappa = -1, \\ r & \text{if } \kappa = 0, \\ R\sin^{-1}(r/R) & \text{if } \kappa = +1. \end{cases} \tag{6}$$

Inverting these relations, we obtain

$$r = \begin{cases} R\sinh(\chi/R) & \text{if } \kappa = -1, \\ \chi & \text{if } \kappa = 0, \\ R\sin(\chi/R) & \text{if } \kappa = +1. \end{cases} \tag{7}$$

Let's now consider the three geometries in turn:

- **Flat (Euclidean) geometry** ($\kappa = 0$): In this case, the metric is that given in Eq. (2) above. This geometry is characterized by:
  1. The inner angles of a triangle add up to π.
  2. The circumference of a circle of radius r is $2\pi r$.
  3. Two lines that are initially parallel will stay parallel forever.

- **Closed (spherical) geometry** ($\kappa = +1$): In this case, the metric takes the form

$$ds^2 = -dt^2 + a^2(t)\left[d\chi^2 + R^2\sin^2(\chi/R)d\Omega^2\right], \tag{8}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. Defining $q \equiv \chi/R$, this can be rewritten as

$$ds^2 = -dt^2 + a^2(t)R^2\left[dq^2 + \sin^2 q d\Omega^2\right], \tag{9}$$

whose spatial part is the metric of a three-sphere of radius $a(t)R$. Since it is difficult to visualize a three-sphere, let's focus on its equatorial plane by setting $\theta = \pi/2$. At a fixed coordinate time t, we then have

$$ds^2 = (aR)^2(dq^2 + \sin^2 q d\phi^2), \tag{10}$$

which the familiar metric of a two-sphere of radius aR. We can then think of q as the "polar" angle of this sphere. Drawing a circle on this sphere at a constant polar angle q, the radius of this circle (as measured along the surface of the sphere) is $r = aRq$. However, the circumference of this circle is $2\pi aR\sin q < 2\pi r$. This is not Euclidean geometry, and this space is said to be (positively) *curved*, with radius of curvature R.

Looking at Eq. (8), we see that when $\chi \ll R$, the metric appears appears approximately flat since $R^2\sin^2(\chi/R) \to \chi^2$ in this case. Only when χ becomes a sizable fraction of the radius of curvature R that we start "feeling" the curvature. This makes sense for us living on Earth: we don't really see that the Earth is roughly round in our everyday lives since we only see a very small area of it at any given time. But if you go to space and see the whole planet at once, it is easy to see it is roughly a sphere.

In cosmology, we refer to this case with positive curvature as an *closed universe*, since at any given time, the Universe is a giant three-sphere and thus have a finite volume. This geometry is is characterized by:

1. The inner angles of a triangle add up to more than π.
2. The circumference of a circle of radius r is less than $2\pi r$.
3. Two lines that are initially parallel will eventually converge.

- **Open (hyperbolic) geometry** ($\kappa = -1$): In this case, the metric takes the form

$$ds^2 = -dt^2 + a^2(t)\left[d\chi^2 + R^2\sinh^2(\chi/R)d\Omega^2\right], \tag{11}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. The spatial part of this metric represents a three-dimensional hyperboloid (saddle) with constant negative curvature $-R$. Much of our discussion for the spherical case applies here, except with the substitution $\sin \to \sinh$. In particular, the circumference of a circle of radius $r = aRq$ (as measured along the saddle; $q = \chi/R$ as before) is $2\pi aR\sinh q > 2\pi r$.

In cosmology, we refer to this case with negative curvature as an *open universe*, since at any given time, the Universe is infinite. This geometry is is characterized by:

1. The inner angles of a triangle add up to less than π.
2. The circumference of a circle of radius r is more than $2\pi r$.
3. Two lines that are initially parallel always diverge from each other.

II. CONNECTION TO FRIEDMANN EQUATION

Having discussed the three possible geometries, we can now go back to Eq. (3) written of in terms of k

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]. \tag{12}$$

It turns out that the k appearing in this metric is the same as that we saw showing up in the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho(t) - \frac{k}{a^2}, \tag{13}$$

where $\rho(t)$ is total energy density of the Universe (including all its components), and where we now understand the terms proportional to k as telling us about the spatial curvature of the Universe. Cosmologists sometime like to define a "curvature" energy density

$$\rho_K(t) \equiv -\frac{3k}{8\pi G a^2}, \tag{14}$$

which allows us to write the Friedmann equation as simply

$$H^2 = \frac{8\pi G}{3} \sum_i \rho_i(t), \tag{15}$$

where the sum over i now includes the curvature term. One can think of the curvature energy density as the "energy cost" to deform flat space into a curved space. One only becomes sensitive to spatial curvature once the linear size of the observable Universe becomes of the order of the radius of curvature. Since $\rho_K \propto 1/a^2$, it always eventually dominates over the energy density of matter ($\propto 1/a^3$) and radiation ($\propto 1/a^4$).