PHYS480/581 Cosmology
Distances in an Expanding Universe II

(Dated: September 21, 2022)

I. KEY DISTANCE THAT IS NOT OBSERVABLE: THE COMOVING DISTANCE

In an expanding Universe like ours, there is a distance that is very important for our understanding of the Universe as a whole, despite not being directly observable. This distance involves the propagation of photons (or any other particle traveling at the speed of light), and therefore require us to integrate null paths ($ds^2 = 0$) using the general FLRW metric

$$ds^2 = -dt^2 + a(t)^2 \left[d\chi^2 + S_k^2(\chi) d\Omega^2\right], \tag{1}$$

where the spatial coordinates are (as usual) comoving, and where

$$S_k(\chi) = \begin{cases} R \sinh(\chi/R) & \text{if } \kappa = -1, \\ \chi & \text{if } \kappa = 0, \\ R \sin(\chi/R) & \text{if } \kappa = +1. \end{cases} \tag{2}$$

A reminder that we have written the curvature constant $k = \kappa/R^2$, with $\kappa = \{-1, 0, 1\}$ and R is the radius of curvature. Since we have

$$\Omega_K = -\frac{k}{H_0^2} = -\frac{\kappa}{R^2 H_0^2}, \tag{3}$$

we have the general relation that

$$R = \frac{1}{\sqrt{|\Omega_K|} H_0}. \tag{4}$$

The comoving distance travelled by a photon emitted at time $t(a)$ (when the scale factor was a) and today is

$$\chi(a) = \int_{t(a)}^{t_0} \frac{dt'}{a(t')} = \int_a^1 \frac{da'}{a'^2 H(a')} = \int_0^{z(a)} \frac{dz'}{H(z')}, \tag{5}$$

where we have used $ds^2 = 0$ with the above metric, assuming pure radial (χ) motion for the photon. While this distance is not directly observable, it is nonetheless very important as other distances are directly related to it. This distance is usually simply referred to as the *comoving distance* to redshift z.

II. KEY DISTANCES THAT ARE OBSERVABLE

A. Angular diameter distance

One way to determine distance in an expanding universe is to measure the angular size θ of an object of *known physical size* l. Assuming this angular size to be small, the distance to that object is defined as

$$d_{\rm A} \equiv \frac{l}{\theta}, \tag{6}$$

where $d_{\rm A}$ is called the *angular diameter distance*. To compute $d_{\rm A}$, we need to relate l to a physical distance computed using the metric from Eq. (1). At an instant in time ($dt = 0$), the physical size of an object sustaining an infinitesimal angular size $d\theta$ is

$$ds = a(t) S_k(\chi) d\theta. \tag{7}$$

For a finite angular size θ, we thus have

$$l = a(t) S_k(\chi) \theta, \tag{8}$$

where $a(t)$ here is the scale factor at which the light we are using to measure the angle θ was emitted. Substituting this in Eq. (6), we obtain

$$d_{\mathrm{A}} = aS_k(\chi) = \begin{cases} \frac{a}{\sqrt{\Omega_K}H_0}\sinh\left(\sqrt{\Omega_K}H_0\chi\right) & \text{if } \Omega_K > 0, \\ a\chi & \text{if } \Omega_K = 0, \\ \frac{a}{\sqrt{|\Omega_K|}H_0}\sin\left(\sqrt{|\Omega_K|}H_0\chi\right) & \text{if } \Omega_K < 0. \end{cases} \tag{9}$$

In the above, the comoving distance χ is given by Eq. (5). In the spatially flat case ($\Omega_K = 0$), the angular diameter distance is simply a rescaled version of the comoving distance

$$d_{\mathrm{A}}^{\mathrm{flat}}(z) = a\chi(z) = \frac{\chi(z)}{1+z}. \tag{10}$$

The angular diameter distance is allows one to turn measurements of angles and redshifts into actual distance measurements.